

JONES-WASSERMANN SUBFACTORS FOR DISCONNECTED INTERVALS

FENG XU

ABSTRACT. We show that the Jones-Wassermann subfactors for disconnected intervals, which are constructed from the representations of loop groups of type A , are finite-depth subfactors. The index value and the dual principal graphs of these subfactors are completely determined. The square root of the index value in the case of two disjoint intervals for vacuum representation is the same as the Quantum 3-manifold invariant of type A evaluated on $S^1 \times S^2$.

§0. INTRODUCTION

Let G be a simply connected simple compact Lie group and let $e \in G$ be the identity element of G . We denote by LG the group of smooth maps from S^1 to G with pointwise multiplication. Let us choose a subset I of S^1 , $I = \bigcup_{i=1}^n I_i$, $\bar{I} \subsetneq S^1$, $\bar{I}_i \cap \bar{I}_j = \emptyset$ for $i \neq j$, $1 \leq i, j \leq n$, and I_i is a connected open subset of S^1 with nonempty interior and the symbol \bar{J} means the closure of such a set J in S^1 . We shall call such an I a $n-1$ -disconnected interval. Notice a 0-disconnected interval is really a connected interval of S^1 such that its complement in S^1 has nonempty interior by our terminology.

Denote by $L_I G = \{g \in LG | g|_{I^c} = e\}$ where I^c denotes the complement of I in S^1 .

LG has an interesting series of projective positive energy representations (see [PS]). We denote such an irreducible representation by π . Then it follows (see [W2] or [Fro]) that $\pi(L_I G)''$, $\pi(L_{I^c} G)'$ are both hyperfinite III_1 factors, and $\pi(L_I G)'' \subset \pi(L_{I^c} G)'$ is irreducible. The inclusion $\pi(L_I G)'' \subset \pi(L_{I^c} G)'$ is called the Jones-Wassermann subfactor.

In his remarkable paper [W2], Antony Wassermann has studied the above inclusion in the case when I is a 0-disconnected interval of S^1 . He shows that when G is of type A all the subfactors above are finite-depth subfactors. When I is a n -disconnected interval with $n \geq 1$, the nature of Jones-Wassermann subfactors $\pi(L_I G)'' \subset \pi(L_{I^c} G)'$ is not clear. In fact, such a question is raised in [W1].

I'd like to thank Professor S.Popa and Professor M.Takesaki for very helpful discussions. This work is partially supported by NSF grant DMS-9500882.

In this paper, we will prove in the case when G is of type A , the inclusion $\pi(L_I G)'' \subset \pi(L_{I^c} G)'$ is of finite depth for any n -disconnected interval I . We will also determine the relevant ring structure which completely determine the dual principle graphs of the inclusion $\pi(L_I G)'' \subset \pi(L_{I^c} G)'$.

Our ideas are very simple and let us explain them briefly here (for details, see §3). For simplicity, let us assume $I = I_1 \cup I_2$ is a 1-disconnected interval. Let $G \subseteq K$ be a conformal pair (see §3) and π be an irreducible positive energy representation of LK of level 1 (See §3.1), then it follows from [W2, W3] that

$$\pi(L_{I_i} G)'' \subset \pi(L_{I_i} K)'' \quad (i = 1, 2)$$

are of finite depth. Moreover $\pi(L_I G)'' \subset \pi(L_I K)''$ is conjugate to (see [B] or [W3]):

$$\pi(L_{I_1} G)'' \hat{\otimes} \pi(L_{I_2} G)'' \subset \pi(L_{I_1} K)'' \hat{\otimes} \pi(L_{I_2} K)''$$

where $\hat{\otimes}$ is the tensor products of von Neumann algebras. It follows that $\pi(L_I G)'' \subset \pi(L_I K)''$ is of finite depth. We then do the following analogue of basic construction of V. Jones:

$$\pi(L_I G)'' \subset \pi(L_I K)'' \subset \pi(L_{I^c} K)' \subset \pi(L_{I^c} G)'$$

It is then easy to see that $\pi(L_I G)'' \subset \pi(L_{I^c} G)'$ has finite index if $\pi(L_I K)'' \subset \pi(L_{I^c} K)'$ has finite index. Moreover, we can choose certain conformal inclusions such that all the Jones-Wassermann subfactors for disjoint intervals appear as the reduced subfactor of $\pi(L_I G)'' \subset \pi(L_{I^c} G)'$. So the question about the finiteness of the index of Jones-Wassermann subfactors associated to LG is reduced to that of LK with representation π of level 1.

Recall when K is a classical Lie group, the level 1 representations π of LK are built from the theory of free fermions. Hence the question about the index of $\pi(L_I K)'' \subset \pi(L_{I^c} K)'$ can be answered by the theory of free fermions which is more tractable.

For our cases, we have considered two conformal inclusions:

$$\begin{aligned} LSU(n) \times LSU(m) &\subset LSU(mn) \\ LU(1) \times LSU(n) &\subset LU(n). \end{aligned}$$

We will show that $\pi(L_I LSU(n))'' \subset \pi(L_{I^c} LSU(n))'$ has finite index and we will obtain an estimation on the index value by the ideas outlined above.

It turns out that, by rather simple computations of the relevant ring structure, we can completely determine the value of index and the dual principal graphs of Jones-Wassermann subfactors for disjoint intervals.

It is worth mentioning that the square root of the index for 1-disconnected interval and associated with the vacuum representation is equal to $\tau(S^2 \times S^1)$ where τ is the 3-manifold invariant as constructed in [Tu]. See §3.5 for details.

Our constructions are very general and apply to other classical Simple Lie groups once the theory of the corresponding connected interval case of Jones-Wassermann

subfactors is established. Such a theory has been outlined in [W2], but details have not yet appeared.

This paper is organized as follows: §1 is a preliminary section of general theory of sectors, correspondences and constructive conformal field theories. §1.2 and §1.3 are contained in [GL1] and we have included them to set up the notations and concepts. In §1.4, we proved Yang-Baxter-Equation (YBE) and Braiding-Fusion-Equation (BFE). We also give a proof monodromy-equations. These results are scattered in the literature and their proof is not new. We have included them for future references. For an example, these equations are used in [X] to obtain a family of braided endomorphisms from conformal inclusions.

In §2 we briefly discuss the representations of loop group following [PS].

In §2.1, the basic representation of $LU(n)$ is introduced, and its decomposition under $LU(1) \times LSU(n)$ is described. The well-known Araki-duality in this context will play an important role in §3.

We sketch some results of [W2] in §2.2, together with similar but much simpler results about $LU(1)$ when the level is even. The local factorization properties are studied in §2.3 following [B] and [W3].

In §3.1 we consider two conformal inclusions:

$$\begin{aligned} LSU(n) \times LSU(m) &\subset LSU(nm), \\ LU(1) \times LSU(n) &\subset LU(n). \end{aligned}$$

. Proposition 3.1.1 determines the index of certain subfactors associated with the above conformal inclusions. We studied the ring structure associated with the Jones-Wassermann subfactors in §3.2. In §3.3, we studied representations of $LU(1)$ at odd level which is important for our purposes. The new feature here is that there is no local structure and instead of Haag duality we have twisted Haag duality. However, we manage to imitate the result in §3.2 to give a crucial estimation of index value of Jones-Wassermann subfactors in Proposition 3.3.2. In §3.4, we proved a special case of our main Theorem 3.5. By using the results of §3.1 to §3.4, together with Araki-duality in §2.1 and the local factorization properties in §2.3, we give the proof of our main Theorem 3.5 in §3.5.

In §4 we give our conclusions and suggest some further questions.

Let us say a few words about notations and terminology.

In this paper, by an *interval* we shall always mean an open connected subset I of S^1 such that I and the interior I^c of its complement are non-empty. We use \mathcal{I} to denote the set of such intervals. Notice the definition of a $n-1$ -disconnected interval is given at the beginning of the introduction. For any inclusion $N \subset M$ of factors, we shall use $d(N \subset M)$ to denote its statistical dimension as defined in [L2].

We shall use $\mathcal{L}G$ to denote central extensions of LG and the specific central extension should be clear from the context.

Let $\pi : \mathcal{L}G \rightarrow LG$ be the canonical map. $L_I G$ is defined to be those elements of LG which are equal to identity element of G on I^c . $\mathcal{L}_I G$ is defined to be $\pi^{-1}(L_I G)$. We shall use \mathbf{G} to denote the universal covering group of $PSL(2, \mathbf{R})$.

§1. PRELIMINARIES

1.1. Sectors and Correspondences.

Let M, N be von Neumann algebras, that we always assume to have separable preduals, and \mathcal{H} a $M - N$ correspondence, namely \mathcal{H} is a (separable) Hilbert space, where M acts on the left, N acts on the right and the actions are normal.

We denote by $x\xi y$, $x \in M$, $y \in N$, $\xi \in \mathcal{H}$ the relative actions.

The trivial $M - M$ correspondence is the Hilbert space $L^2(M)$ with the standard actions given by the modular theory

$$x\xi y = xJy^*J\xi, \quad x, y \in M, \quad \xi \in L^2(M),$$

where J is the modular conjugation of M ; the unitary correspondence is well defined modulo unitary equivalence.

If ρ is a normal homomorphism of M into M we let AH_ρ be the Hilbert space $L^2(M)$ with actions: $x \cdot \xi \cdot y \equiv \rho(x)\xi \cdot y$, $x \in M$, $y \in M$, $\xi \in L^2(M)$. Denote by $\text{End}(M)$ the semigroup of the endomorphism of M and $\text{Corr}(M)$ the set of all $M - M$ correspondences. The following proposition is proved in [L4] (Corollary 2.2 in [L4]).

Proposition 1.1. *Let M be an infinite factor. There exists a bijection between $\text{End}(M)$ and $\text{Corr}(M)$. Given $\rho, \rho' \in \text{End}(M)$, H_ρ is equivalent to $H_{\rho'}$ iff there exists a unitary $u \in M$ with $\rho'(x) = u\rho(x)u^*$.*

Let $\text{Sect}(M)$ denote the quotient of $\text{End}(M)$ modulo unitary equivalence in M as in Proposition 1.1. We call sectors the elements of the semigroup $\text{Sect}(M)$; if $\rho \in \text{End}(M)$ we denote by $[\rho]$ its class in $\text{Sect}(M)$. By Proposition 2.2 $\text{Sect}(M)$ may be naturally identified with $\text{Corr}(M)^\sim$ the quotient of $\text{Corr}(M)$ modulo unitary equivalence. It follows from [L3] and [L4] that $\text{Sect}(M)$, with M a properly infinite (on Hilbert space \mathcal{H}) von Neumann algebra, is endowed with a natural involution $\theta \rightarrow \bar{\theta}$ that commutes with all natural operations of direct sum, tensor product and other (the tensor product of correspondences correspond to the composition of sectors). Denote by $\text{Sect}_0(M)$ those elements of $\text{Sect}(M)$ with finite statistical dimensions. For $\lambda, \mu \in \text{Sect}_0(M)$, let $\text{Hom}(\lambda, \mu)$ denote the space of intertwiners from λ to μ , i.e. $a \in \text{Hom}(\lambda, \mu)$ iff $a\lambda(x) = \mu(x)a$ for any $x \in M$. $\text{Hom}(\lambda, \mu)$ is a finite dimensional vector space and we use $\langle \lambda, \mu \rangle$ to denote the dimension of this space. $\langle \lambda, \mu \rangle$ depends only on $[\lambda]$ and $[\mu]$. Moreover we have $\langle \nu\lambda, \mu \rangle = \langle \lambda, \bar{\nu}\mu \rangle$, $\langle \nu\lambda, \mu \rangle = \langle \lambda, \mu\lambda \rangle$ which follows from Frobenius duality (See [L2] or [Y]). We will also use the following notations: if μ is a subsector of λ , we will write as $\mu \prec \lambda$ or $\lambda \succ \mu$.

1.2. General properties of conformal precosheaves on S^1 .

In this section we recall the basic properties enjoyed by the family of the von Neumann algebras associated with a conformal Quantum Field Theory on S^1 . All the propositions in this section and §1.3 are proved in [GL1].

By an *interval* in this section only we shall always mean an open connected subset I of S^1 such that I and the interior I' of its complement are non-empty. We shall denote by \mathcal{I} the set of intervals in S^1 .

A precosheaf \mathcal{A} of von Neumann algebras on the intervals of S^1 is a map

$$I \rightarrow \mathcal{A}(I)$$

from \mathcal{I} to the von Neumann algebras on a Hilbert space \mathcal{H} that verifies the following property:

A. Isotony. If I_1, I_2 are intervals and $I_1 \subset I_2$, then

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2).$$

\mathcal{A} is a conformal precosheaf of von Neumann algebras if the following properties B-E hold too.

B. Conformal invariance. There is a unitary representation U of \mathbf{G} (the universal covering group of $PSL(2, \mathbf{R})$) on \mathcal{H} such that

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI), \quad g \in \mathbf{G}, \quad I \in \mathcal{I}.$$

The group $PSL(2, \mathbf{R})$ is identified with the Möbius group of S^1 , i.e. the group of conformal transformations on the complex plane that preserve the orientation and leave the unit circle globally invariant. Therefore \mathbf{G} has a natural action on S^1 .

C. Positivity of the energy. The generator of the rotation subgroup $U(R)(\cdot)$ is positive.

Here $R(\vartheta)$ denotes the (lifting to \mathbf{G} of the) rotation by an angle ϑ . In the following we shall often write $U(\vartheta)$ instead of $U(R(\vartheta))$. We may associate two one-parameter groups with any interval I . Let I_1 be the upper semi-circle, i.e. the interval $\{e^{i\vartheta}, \vartheta \in (0, \pi)\}$. We identify this interval with the positive real line \mathbf{R}_+ via the Cayley transform $C : S^1 \rightarrow \mathbf{R} \cup \{\infty\}$ given by $z \rightarrow -i(z+1)^{-1}$. Then we consider the one-parameter groups $\Lambda_{I_1}(s)$ and $T_{I_1}(t)$ of diffeomorphisms of S^1 (cf. Appendix B of [GL1]) such that

$$C\Lambda_{I_1}(s)C^{-1}x = e^s x, \quad CT_{I_1}(t)C^{-1}x = x + t, \quad t, s, x \in \mathbf{R}.$$

We also associate with I_1 the reflection r_{I_1} given by

$$r_{I_1}z = \bar{z}$$

where \bar{z} is the complex conjugate of z . We remark that Λ_{I_1} restricts to an orientation preserving diffeomorphisms of I_1 , r_{I_1} restricts to an orientation reversing diffeomorphism of I_1 onto I_1' and $T_{I_1}(t)$ is an orientation preserving diffeomorphism of I_1 into itself if $t \geq 0$.

Then, if I is an interval and we chose $g \in \mathbf{G}$ such that $I = gI_1$ we may set

$$\Lambda_I = g\Lambda_{I_1}g^{-1}, \quad r_I = gr_{I_1}g^{-1}, \quad T_I = gT_{I_1}g^{-1}.$$

The elements $\Lambda_I(s)$, $s \in \mathbf{R}$ and r_I are well defined, while the one parameter group T_I is defined up to a scaling of the parameter. However, such a scaling plays no role in this paper. We note also that $T_{I'}(t)$ is an orientation preserving diffeomorphism of I into itself if $t \leq 0$.

D. Locality. If I_0, I are disjoint intervals then $\mathcal{A}(I_0)$ and $\mathcal{A}(I)$ commute.

The lattice symbol \vee will denote ‘the von Neumann algebra generated by’.

E. Existence of the vacuum. There exists a unit vector Ω (vacuum vector) which is $U(\mathbf{G})$ -invariant and cyclic for $\vee_{I \in \mathcal{I}} \mathcal{A}(I)$.

Let r be an orientation reversing isometry of S^1 with $r^2 = 1$ (e.g. r_{I_1}). The action of r on $PSL(2, \mathbf{R})$ by conjugation lifts to an action σ_r on \mathbf{G} , therefore we may consider the semidirect product of $\mathbf{G} \times_{\sigma_r} \mathbf{Z}_2$. Any involutive orientation reversing isometry has the form $R(\vartheta)r_{I_1}R(-\vartheta)$, thus $\mathbf{G} \times_{\sigma_r} \mathbf{Z}_2$ does not depend on the particular choice of the isometry r . Since $\mathbf{G} \times_{\sigma_r} \mathbf{Z}_2$ is a covering of the group generated by $PSL(2, \mathbf{R})$ and r , $\mathbf{G} \times_{\sigma_r} \mathbf{Z}_2$ acts on S^1 . We call (anti-)unitary a representation U of $\mathbf{G} \times_{\sigma_r} \mathbf{Z}_2$ by operators on \mathcal{H} such that $U(g)$ is unitary, resp. antiunitary, when g is orientation preserving, resp. orientation reversing. Then we have the following (See Prop.1.1 of [GL1]):

1.2.1 Proposition. *Let \mathcal{A} be a conformal precosheaf. The following hold:*

- (a) *Reeh-Schlieder theorem: Ω is cyclic and separating for each von Neumann algebra $\mathcal{A}(I)$, $I \in \mathcal{I}$.*
- (b) *Bisognano-Wichmann property: U extends to an (anti-)unitary representation of $\mathbf{G} \times_{\sigma_r} \mathbf{Z}_2$ such that, for any $I \in \mathcal{I}$,*

$$U(\Lambda_I(2\pi t)) = \Delta_I^{it} \\ U(r_I) = J_I$$

where Δ_I, J_I are the modular operator and the modular conjugation associated with $(\mathcal{A}(I), \Omega)$ [29]. For each $g \in \mathbf{G} \times_{\sigma_r} \mathbf{Z}_2$

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI).$$

- (c) *Additivity: if a family of intervals I_i covers the interval I , then*

$$\mathcal{A}(I) \subset \vee_i \mathcal{A}(I_i).$$

- (d) *Spin and statistics for the vacuum sector [16]: U is indeed a representation of $PSL(2, \mathbf{R})$, i.e. $U(2\pi) = 1$.*
- (e) *Haag duality:*

F. Uniqueness of the vacuum (or irreducibility). The only $U(\mathbf{G})$ -invariant vectors are the scalar multiples of Ω .

The term irreducibility is due to the following (See Prop.1.2 of [GL1]):

1.2.2 Proposition. *The following are equivalent:*

- (i) $C\Omega$ are the only $U(\mathbf{G})$ -invariant vectors.
- (ii) The algebras $\mathcal{A}(I)$, $I \in \mathcal{I}$, are factors. In this case they are type III_1 factors.
- (iii) If a family of intervals I_i intersects at only one point ζ , then $\cap_i \mathcal{A}(I_i) = \mathbf{C}$.
- (iv) The von Neumann algebra $\vee \mathcal{A}(I)$ generated by the local algebra coincides with $\mathcal{B}(\mathcal{H})$ (\mathcal{A} is irreducible).

Now any conformal precosheaf decomposes uniquely into a direct integral of irreducible conformal precosheaves. This can be seen as in Proposition 3.1 of [GL3]. We will therefore always assume that our precosheaves are irreducible.

1.3. Superselection structure..

In this section \mathcal{A} is an irreducible conformal precosheaf of von Neumann algebras as defined in Section 1.2.

A covariant *representation* π of \mathcal{A} is a family of representations π_I of the von Neumann algebras $\mathcal{A}(I)$, $I \in \mathcal{I}$, on a Hilbert space \mathcal{H}_π and a unitary representation U_π of the covering group \mathbf{G} of $PSL(2, \mathbf{R})$, with *positive energy*, i.e. the generator of the rotation unitary subgroup has positive generator, such that the following properties hold:

$$\begin{aligned} I \supset \bar{I} &\Rightarrow \pi_{\bar{I}}|_{\mathcal{A}(I)} = \pi_I \quad (\text{isotony}) \\ \text{ad}U_\pi(g) \cdot \pi_I &= \pi_{gI} \cdot \text{ad}U(g) \quad (\text{covariance}). \end{aligned}$$

A unitary equivalence class of representations of \mathcal{A} is called *superselection sector*.

Assuming \mathcal{H}_π to be separable, the representations π_I are normal because the $\mathcal{A}(I)$'s are factors. Therefore for any given I_0 , $\pi_{I'_0}$ is unitarily equivalent to $\text{id}_{\mathcal{A}(I'_0)}$ because $\mathcal{A}(I'_0)$ is a type III factor. By identifying \mathcal{H}_π and \mathcal{H} , we can thus assume that π is localized in a given interval $I_0 \in \mathcal{I}$, i.e. $\pi_{I'_0} = \text{id}_{\mathcal{A}(I'_0)}$ (cf. [Fro]). By Haag duality we then have $\pi_I(\mathcal{A}(I)) \subset \mathcal{A}(I)$ if $I \supset I_0$. In other words, given $I_0 \in \mathcal{I}$ we can choose in the same sector of π a *localized endomorphism* with localization support in I_0 , namely a representation ρ equivalent to π such that

$$I \in \mathcal{I}, I \supset I_0 \Rightarrow \rho_I \in \text{End } \mathcal{A}(I), \quad \rho_{I'_0} = \text{id}_{I'_0}.$$

In the following representations are always assumed to be covariant with positive energy.

To capture the global point of view we may consider the *universal algebra* $C^*(\mathcal{A})$. Recall that $C^*(\mathcal{A})$ is a C^* -algebra canonically associated with the precosheaf \mathcal{A} (see [Fre]). There are injective embeddings $\iota_I : \mathcal{A}(I) \rightarrow C^*(\mathcal{A})$ so that the local von Neumann algebras $\mathcal{A}(I)$, $I \in \mathcal{I}$, are identified with subalgebras of $C^*(\mathcal{A})$ and generate all together a dense $*$ -subalgebra of $C^*(\mathcal{A})$, and every representation of the precosheaf \mathcal{A} factors through a representation of $C^*(\mathcal{A})$. Conversely any representation of $C^*(\mathcal{A})$ restricts to a representation of \mathcal{A} . The vacuum representation π_0 of $C^*(\mathcal{A})$ corresponds to the identity representation of \mathcal{A} on \mathcal{H} , thus π_0 acts

identically on the local von Neumann algebras. We shall often drop the symbols ι_I and π_0 when no confusion arises.

By the universality property, for each $g \in PSL(2, \mathbf{R})$ the isomorphism $\text{ad}U(g) : \mathcal{A}(I) \rightarrow \mathcal{A}(gI)$, $I \in \mathcal{I}$ lifts to an automorphism α_g of $C^*(\mathcal{A})$. It will be convenient to lift the map $g \rightarrow \alpha_g$ to a representation, still denoted by α , of the universal covering group \mathbf{G} of $PSL(2, \mathbf{R})$ by automorphisms of $C^*(\mathcal{A})$.

The covariance property for an endomorphism ρ of $C^*(\mathcal{A})$ localized in I_0 means that $\alpha_g \cdot \rho \cdot \alpha_{g^{-1}}$ is

$$\text{ad}z_\rho(g)^* \cdot \rho = \alpha_g \cdot \rho \cdot \alpha_{g^{-1}} \quad g \in \mathbf{G}$$

for a suitable unitary $z_\rho(g) \in C^*(\mathcal{A})$. We define

$$\rho_g = \alpha_g \cdot \rho \cdot \alpha_{g^{-1}} \quad , g \in \mathbf{G}$$

$\rho_{g,J}$ is the restriction of ρ_g to $\mathcal{A}(J)$. The map $g \rightarrow z_\rho(g)$ can be chosen to be a localized α -cocycle, i.e.

$$\begin{aligned} z_\rho(g) &\in \mathcal{A}(I_0 \cup gI_0) \quad \forall g \in \mathbf{G} : I_0 \cup gI_0 \in \mathcal{I} \\ z_\rho(gh) &= z_\rho(g)\alpha_g(z_\rho(h)), \quad g, h \in \mathbf{G}. \end{aligned}$$

The relations between (π, U_π) and (ρ, z_ρ) are

$$\begin{aligned} \pi &= \pi_0 \cdot \rho \\ \pi_0(z_\rho(g)) &= U_\pi(g)U(g)^* \end{aligned}$$

To compare with the result of [Fro], let us define:

$$\Gamma_\rho(g) = \pi_0(z_\rho(g)^*)$$

As is known (see [GL2]) that the localized cocycle z_ρ reconstructs the endomorphism ρ via

$$\rho|_{\mathcal{A}(gI'_0)} = \text{ad}z_\rho(g)|_{\mathcal{A}(gI'_0)}$$

A localized endomorphism of $C^*(\mathcal{A})$ is said *irreducible* if the associated representation π is irreducible.

Note that the representations $\pi_0 \cdot \rho_1$ and $\pi_0 \cdot \rho_2$ associated with the endomorphisms ρ_1, ρ_2 of $C^*(\mathcal{A})$ are unitarily equivalent if and only if ρ_1 and ρ_2 are equivalent endomorphisms of \mathcal{A} , i.e. ρ_2 is a perturbation of ρ_1 by an inner automorphism of \mathcal{A} .

An endomorphism of $C^*(\mathcal{A})$ localized in an interval I_0 is said to have *finite index* if $\rho_I (= \rho|_{\mathcal{A}(I)})$ has finite index, $I_0 \subset I$ (see [L2,L3]). The index is indeed well defined due to the following (See Prop.2.1 of [GL1])

1.3.1 Proposition. *Let ρ be an endomorphism localized in the interval I_0 . Then the index $\text{Ind}(\rho) := \text{Ind}(\rho_I)$, the minimal index of ρ_I , does not depend on the interval $I \supset I_0$.*

The following Proposition is Prop.2.2 of [GL1]:

1.3.2 Proposition. *Let ρ be a covariant (not necessarily irreducible) endomorphism with finite index. Then the representation U_ρ described before is unique. In particular, any irreducible component of ρ is a covariant endomorphism.*

By the above proposition the *univalence* of an endomorphism ρ is well defined by

$$S_\rho = U_\rho(2\pi).$$

By definition S_ρ belongs to $\pi(C^*(\mathcal{A}))'$ therefore, when ρ is irreducible, S_ρ is complex number of modulus one

$$S_\rho = e^{2\pi i L_\rho}$$

with L_ρ the lowest weight of U_ρ . In this case, since $U_{\rho'}(g) := \pi_0(u)U_\rho(g)\pi_0(u)^*$, where $\rho'(\cdot) := u\rho(\cdot)u^*$, $u \in C^*(\mathcal{A})$, then S_ρ depends only on the superselection class of ρ .

Let ρ_1, ρ_2 be endomorphisms of an algebra \mathcal{B} . Their intertwiner space is defined by

$$(\rho_1, \rho_2) = \{T \in \mathcal{B} : \rho_2(x)T = T\rho_1(x), \quad x \in \mathcal{B}\}$$

In case $\mathcal{B} = C^*(\mathcal{A})$, ρ_i localized in the interval I_i and $T \in (\rho_1, \rho_2)$, then $\pi_0(T)$ is an intertwiner between the representations $\pi_0 \cdot \rho_i$. If $I \supset I_1 \cup I_2$, then by Haag duality its embedding $\iota_I \cdot \pi_0(T)$ is still an intertwiner in (ρ_1, ρ_2) and a local operator. We shall denote by $(\rho_1, \rho_2)_I$ the space of such local intertwiners

$$(\rho_1, \rho_2)_I = (\rho_1, \rho_2) \cap \mathcal{A}(I).$$

If I_1 and I_2 are disjoint, we may cover $I_1 \cup I_2$ by an interval I in two ways: we adopt the convention that, unless otherwise specified, a local *intertwiner* is an element of $(\rho_1, \rho_2)_I$ where I_2 follows I_1 inside I in the clockwise sense.

We now define the statistics. Given the endomorphism ρ of \mathcal{A} localized in $I \in \mathcal{I}$, choose an equivalent endomorphism ρ_0 localized in an interval $I_0 \in \mathcal{I}$ with $\bar{I}_0 \cap \bar{I} = \emptyset$ and let u be a local intertwiner in (ρ, ρ_0) as above, namely $u \in (\rho, \rho_0)_{\bar{I}}$ with I_0 following clockwise I inside \bar{I} .

The *statistics operator* $\varepsilon := u^*\rho(u) = u^*\rho_{\bar{I}}(u)$ belongs to (ρ_I^2, ρ_I^2) . An elementary computation shows that it gives rise to a presentation of the Artin braid group

$$\varepsilon_i \varepsilon_{i+1} \varepsilon_i = \varepsilon_{i+1} \varepsilon_i \varepsilon_{i+1}, \quad \varepsilon_i \varepsilon_{i'} = \varepsilon_{i'} \varepsilon_i \quad \text{if } |i - i'| \geq 2,$$

where $\varepsilon_i = \rho^{i-1}(\varepsilon)$. The (unitary equivalence class of the) representation of the braid group thus obtained is the *statistics* of the superselection sector ρ .

Recall that if ρ is an endomorphism of a C^* -algebra \mathcal{B} , a *left inverse* of ρ is a completely positive map Φ from \mathcal{B} to itself such that $\Phi \cdot \rho = \text{id}$.

We shall see in Corollary 2.12 that if ρ is irreducible there exists a unique left inverse Φ of ρ and that the *statistics parameter*

$$\lambda_\rho := \Phi(\varepsilon)$$

depends only on the sector of ρ .

The *statistical dimension* $d(\rho)$ and the *statistics phase* κ_ρ are then defined by

$$d(\rho) = |\lambda_\rho|^{-1}, \quad \kappa_\rho = \frac{\lambda_\rho}{|\lambda_\rho|}.$$

In [GL1], κ_ρ is shown to be equal to S_ρ under rather general conditions. But we will not use this relation.

1.4 Coherence equations.

In this section, we assume Δ is a set of localized covariant endomorphism of \mathcal{A} with localization support in I_0 . Let h, g be elements of \mathbf{G} . We assume $hI_0 \cap I_0 = \emptyset$, $gI_0 \cap I_0 = \emptyset$, $hI_0 \cap gI_0 = \emptyset$. Choose $J_1, J_2 \in \mathcal{I}$ such that $J_1 \cup J_2 \subsetneq S^1$, $J_1 \supset I_0 \cup g.I_0$, $J_2 \supset I_0 \cup h.I_0$, $J_1 \cap h.I_0 = \emptyset$, $J_2 \cap g.I_0 = \emptyset$ and $J_1 \cap J_2 = I_0$. We assume in J_1 (resp. J_2), $g.I_0$ (resp. $h.I_0$) lies a clockwise (resp. anti clockwise) from I_0 .

Lemma 1.4.1. *For any $J \supset J_1 \cup J_2$, $J \in \mathcal{I}$, $\gamma \in \Delta$ and $x \in \mathcal{A}(J)$, we have*

- (0) $\Gamma_\lambda(g) \in \mathcal{A}(J_1)$.
- (1) $\Gamma_\lambda(g)^* \gamma_{J_1}(\Gamma_\lambda(g)) \gamma_J \cdot \lambda_J(x) = \lambda_J \cdot \gamma_J(x) \Gamma_\lambda(g)^* \gamma_{J_1}(\Gamma_\lambda(g))$.
- (2) $\Gamma_\lambda(g)^* \gamma_{J_1}(\Gamma_\lambda(g)) = \lambda_{J_2}(\Gamma_\gamma(h)^*) \Gamma_\gamma(h)$
- (3) $\Gamma_\gamma(g)^* \gamma_{J_1}(\Gamma_\lambda(g)) \in \mathcal{A}(I_0)$.

Proof. Recall $\lambda_J(x) = \Gamma_\gamma(g)^* \lambda_{g,J}(x) \Gamma_\lambda(g)$ for any $x \in \mathcal{A}(J)$, $S^1 \supsetneq J \supset J_1$. Since λ_J (resp. $\lambda_{g,J}$) is localized on I_0 (resp. $g.I_0$), it follows that $\Gamma_\lambda(g) \in \mathcal{A}(J \cap J_1^c)'$ for any $S^1 \supsetneq J \supset J_1$. Let us choose $J_2 \supset J_1, J_3 \supset J_1$ so that $I_2 = J_2 \cap J_1^c, I_3 = J_3 \cap J_1^c$ are closed intervals and $I_2^c \cap I_3^c = J_1$. Then we have:

$$\Gamma_\lambda(g) \in \mathcal{A}(I_2^c) \cap \mathcal{A}(I_3^c)$$

We claim that

$$\mathcal{A}(I_2^c) \cap \mathcal{A}(I_3^c) = \mathcal{A}(J_1)$$

In fact it is clear that $\mathcal{A}(I_2^c) \cap \mathcal{A}(I_3^c) \supset \mathcal{A}(J_1)$. By Haag duality, it is sufficient to prove $\mathcal{A}(I_2) \vee \mathcal{A}(I_3) \supset \mathcal{A}(J_1^c)$. But $I_2 \cup I_3 = J_1^c$ and the inclusion above follows by (c) of Prop.1.2.1. So we have $\Gamma_\lambda(g) \in \mathcal{A}(J_1)$.

By (0), $\gamma_{J_1}(\Gamma_\lambda(g))$ is well defined. To prove (1), we can calculate the left hand side as follows:

$$\begin{aligned}
& \Gamma_\lambda(g)^* \gamma_{J_1}(\Gamma_\lambda(g)) \gamma_J \cdot \lambda_J(x) \\
&= \Gamma_\lambda(g)^* \gamma_J(\Gamma_\lambda(g) \lambda_J(x)) \\
&= \Gamma_\lambda(g)^* \gamma_J(\lambda_{g,J}(x) \Gamma_\lambda(g)) \\
&= \Gamma_\lambda(g)^* \gamma_J(\lambda_{g,J}(x)) \gamma_{J_1}(\Gamma_\lambda(g)) \\
&= \Gamma_\lambda(g)^* \lambda_{g,J}(\gamma_J(x)) \gamma_{J_1}(\Gamma_\lambda(g)) \\
&= \lambda_J \cdot \gamma_J(x) \Gamma_\lambda(g)^* \gamma_{J_1}(\Gamma_\lambda(g))
\end{aligned}$$

where in the first “=” we used $\gamma_J(x) = \gamma_{J_1}(x)$ if $x \in \mathcal{A}(J)\mathcal{A}(J_1)$ and $J \supset J_1$. In the fourth “=” we used $\lambda_{g,J}(\gamma_J(x)) = \gamma_J(\lambda_{g,J}(x))$ for $x \in \mathcal{A}(J)$ since $\lambda_{g,J}$ and γ_J have disjoint support.

To prove (2), it is sufficient to prove:

$$\begin{aligned}
& \Gamma_\gamma(h)^* \Gamma_\gamma(h) \gamma_{J_1}(\Gamma_\lambda(g)) \Gamma_\gamma(h)^* \\
&= \Gamma_\lambda(g) \lambda_{J_2}(\Gamma_\gamma(h)^*) \Gamma_\lambda(g)^* \Gamma_\lambda(g)
\end{aligned}$$

i.e.

$$\Gamma_\gamma(h)^* \gamma_{h,J_1}(\Gamma_\lambda(g)) = \lambda_{g,J_2}(\Gamma_\gamma(h)^*) \Gamma_\gamma(g).$$

This follows from

$$\begin{aligned}
& \gamma_{h,J_1}(\Gamma_\lambda(g)) = \Gamma_\lambda(g) \\
& \lambda_{g,J_2}(\Gamma_\gamma(h)^*) = \Gamma_\gamma(h)^*.
\end{aligned}$$

Since $\Gamma_\gamma(g)$ (resp. $\Gamma_\gamma(h)^*$) is in $\mathcal{A}(J_1)$ (resp. $\mathcal{A}(J_2)$) and J_1 (resp. J_2) is disjoint from the support $h.I_0$ (resp. $g.I_0$) of γ_{h,J_1} (resp. λ_{g,J_2}).

It follows from (1) and the proof of (0) that $\Gamma_\lambda(g)^* \gamma_{J_1}(\Gamma_\lambda(g)) \in \mathcal{A}(J_1)$.

Similarly, $\lambda_{J_2}(\Gamma_\gamma(h)^*) \Gamma_\gamma(h) \in \mathcal{A}(J_2)$.

From (2) we deduce that $\Gamma_\lambda(g)^* \gamma_{J_1}(\Gamma_\lambda(g)) \in \mathcal{A}(J_1) \cap \mathcal{A}(J_2) = \mathcal{A}(J_0)$ where the last “=” follows as in the proof of (0). \square

Because the property (1) of Lemma 1.4.1, $\Gamma_\lambda(g)^* \gamma_{J_1}(\Gamma_\lambda(g))$ is called the braiding operator.

For simplicity, we use $\sigma_{\gamma,\lambda}$ to denote $\Gamma_\lambda(g)^* \gamma_{J_1}(\Gamma_\lambda(g))$. We are now ready to prove the following equations. For simplicity we will drop the subscript I_0 and write μ_{I_0} as μ for any $\mu \in \Delta$ in the following.

Proposition 1.4.2. (1) *Yang-Baxter-Equation (YBE)*

$$\sigma_{\mu,\gamma} \mu(\sigma_{\lambda,\gamma}) \sigma_{\lambda,\mu} = \gamma(\sigma_{\lambda,\mu}) \sigma_{\lambda,\gamma} \lambda(\sigma_{\mu,\gamma}).$$

(2) *Braiding-Fusion-Equation (BFE)*

For any $w \in \text{Hom}(\mu\gamma, \delta)$

$$\sigma_{\lambda, \delta} \lambda(w) = w \mu(\sigma_{\lambda, \gamma}) \sigma_{\lambda, \mu} \quad (\text{a})$$

$$\sigma_{\delta, \lambda} w = \lambda(w) \sigma_{\mu, \lambda} \mu(\sigma_{\gamma, \lambda}). \quad (\text{b})$$

Proof. To prove (1), let us first calculate the left handside of (1) as follows:

$$\begin{aligned} & \sigma_{\mu, \gamma} \mu(\sigma_{\lambda, \gamma}) \sigma_{\lambda, \mu} \\ &= \sigma_{\mu, \gamma} \mu(\gamma_{J_2}(\Gamma_\lambda(h)^*) \Gamma_\lambda(h)) \mu_{J_2}(\Gamma_\lambda(h)^*) \Gamma_\gamma(h) \\ &= \sigma_{\mu, \gamma} \mu(\gamma_{J_2}(\Gamma_\lambda(h)^*)) \Gamma_\gamma(h). \end{aligned}$$

For the right hand side of (1), we have:

$$\begin{aligned} & \gamma(\sigma_{\lambda, \mu}) \sigma_{\lambda, \gamma} \lambda(\sigma_{\mu, \gamma}) \\ &= \gamma(\mu_{J_2}(\Gamma_\lambda(h)^*) \Gamma_\lambda(h)) \cdot \gamma_{J_2}(\Gamma_\lambda(h)^*) \Gamma_\lambda(h) \cdot \lambda(\sigma_{\mu, \gamma}) \\ &= \gamma(\mu_{J_2}(\Gamma_\lambda(h)^*)) \lambda_h(\sigma_{\mu, \gamma}) \cdot \Gamma_\lambda(h) \\ &= \gamma(\mu_{J_2}(\Gamma_\lambda(h)^*)) \sigma_{\mu, \gamma} \Gamma_\lambda(h) \\ &= \sigma_{\mu, \gamma} \mu(\gamma_{J_2}(\Gamma_\lambda(h)^*)) \end{aligned}$$

where in the second “=” we have used $\Gamma_\lambda(h) \lambda(\sigma_{\mu, \gamma}) = \lambda_h(\sigma_{\mu, \gamma}) \Gamma_\lambda(h)$; In the third “=” we have used $\lambda_h(\sigma_{\mu, \gamma}) = \sigma_{\mu, \gamma}$ since $\sigma_{\mu, \gamma} \in \mathcal{A}(I_0)$ and λ_h has support on $h \cdot I_0$ which is disjoint from I_0 .

To prove (a) of (2), let us calculate, starting from the right hand side of (a) as follows:

$$\begin{aligned} & w \mu(\sigma_{\lambda, \gamma}) \sigma_{\lambda, \mu} \\ &= w \mu(\gamma_{J_2}(\Gamma_\lambda(h)^*) \Gamma_\lambda(h)) \cdot \mu(\Gamma_\lambda(h)^*) \Gamma_\lambda(h) \\ &= w \mu(\gamma_{J_2}(\Gamma_\lambda(h)^*)) \Gamma_\lambda(h) \\ &= \delta(\Gamma_\lambda(h)^*) w \Gamma_\lambda(h) \\ &= \sigma_{\lambda, \delta} \lambda(w). \end{aligned}$$

To prove (b), we make use of (2) in Lemma 1.4.1 to calculate, starting from the right hand side of (b) in the following:

$$\begin{aligned} & \lambda(w) \sigma_{\mu, \lambda} \mu(\sigma_{\gamma, \lambda}) \\ &= \lambda(w) \Gamma_\lambda(g)^* \mu(\Gamma_\lambda(g)) \cdot \mu(\Gamma_\lambda(g)^* \gamma(\Gamma_\lambda(g))) \\ &= \lambda(w) \Gamma_\lambda(g)^* \mu(\gamma(\Gamma_\lambda(g))) \\ &= \Gamma_\lambda(g)^* w \mu(\gamma(\Gamma_\lambda(g))) \\ &= \Gamma_\lambda(g)^* \delta(\Gamma_\lambda(g)) w = \sigma_{\delta, \lambda} w. \quad \square \end{aligned}$$

Suppose $\xi_1 \in I_{\xi_1} \subset J_1$, $I_{\xi_1} \cap gI_1 \cap I_1 = \emptyset$, $\xi_2 \in I_{\xi_2} \subset J_2$, and $I_{\xi_2} \cap I_1 \cap h.I_1 = \emptyset$. Here g, h, J_1, J_2 are defined as the beginning of this section.

It follows from (2) of Lemma 1.4.1 that:

$$\gamma_{I_{\xi_1}^c}(\Gamma_\lambda(g))^* \Gamma_\lambda(g) = \gamma_{J_2}(\Gamma_\lambda(h)^*) \Gamma_\lambda(h) = \sigma_{\lambda, \gamma}.$$

Hence $\sigma_{\lambda, \gamma} \sigma_{\gamma, \lambda} = \gamma_{I_{\xi_1}^c}(\Gamma_\lambda(g)^*) \gamma_{I_{\xi_2}^c}(\Gamma_\lambda(g))$.

$\sigma_{\lambda, \gamma} \sigma_{\gamma, \lambda}$ is called *monodromy operator*.

Let $T_e : \delta \rightarrow \gamma\lambda$ be an intertwiner.

Recall $S_\rho = U_\rho(2\pi)$ is the univalence of a covariant endomorphism. When ρ is irreducible, S_ρ is a complex number.

Proposition 1.4.3(monodromy equation). *If $S_\delta, S_\gamma, S_\lambda$ are complex numbers, then*

$$T_e^* \gamma_{I_{\xi_1}^c}(\Gamma_\lambda(g)^*) \gamma_{I_{\xi_2}^c}(\Gamma_\lambda(g)) T_e = T_e^* \sigma_{\lambda, \gamma} \sigma_{\gamma, \lambda} T_e = \frac{S_\delta}{S_\lambda S_\gamma}.$$

Proof. The proof is essentially contained in [Boe]. We define

$$W = \{g \in G \mid I_0 \cup gI_0 \text{ is a proper interval contained in } S^1 \setminus I_{\xi_1}\}.$$

For $g \in W$, we define $U_{\gamma\lambda}(g) = \gamma_{I_0 \cup g.I_0}(\Gamma_\lambda(g)) U_\gamma(g)$. Then it is easy to check that if $g_1 \in W, g_2 \in W$ and $g_1 g_2 \in W$, then we have:

$$U_{\gamma\lambda}(g_1 g_2) = U_{\gamma\lambda}(g_1) U_{\gamma\lambda}(g_2).$$

For any $g \in \mathbf{G}$, since \mathbf{G} is connected, g can be decomposed as $g = g_1 \cdots g_n$ with $g_i \in W$. Define

$$U_{\gamma\lambda}(g) = U_{\gamma\lambda}(g_1) \cdots U_{\gamma\lambda}(g_n).$$

By a standard deformation argument, using the fact that G is simply connected, (see the proof of (1) of Proposition 3.3.1 or Proposition 8.2 in [GL2]). $U_{\gamma\lambda}(g)$ is independent of the decomposition of g . It follows from the proof of (v) of Lemma 4.8 in [Fro] that:

$$U_{\gamma\lambda}(g)(\gamma \cdot \lambda)_J(x) U_{\gamma\lambda}(g) = (\gamma \cdot \lambda)_{gJ}(\alpha_{g \cdot x})$$

for any $x \in A_J$.

Since $T_e^* U_{\gamma\lambda}(g) T_e$ is a representation of \mathbf{G} associated with δ , it follows from Proposition 1.3.2 that

$$U_\delta(g) = T_e^* U_{\gamma\lambda}(g) T_e.$$

We may assume, for simplicity, that I_0 is so small that $I_0 \cap \pi.I_0 = \emptyset$. Notice that in particular

$$U_\delta(2\pi) = S_\delta = T_e^* U_{\gamma\lambda}(2\pi) T_e.$$

Choose I_{ξ_1}, I_{ξ_2} such that $I_{\xi_1}, I_{\xi_2}, I_0, \pi.I_0$ don't intersect and anti-clockwise on the circle the order of the intervals are $I_0, I_{\xi_2}, \pi.I_0, I_{\xi_1}$. We have:

$$\begin{aligned}
U_{\gamma\lambda}(2\pi) &= U_{\gamma\lambda}(\pi) \cdot U_{\gamma\lambda}(-\pi)^* \\
&= \gamma_{I_{\xi_1}^c}(\Gamma_\lambda(\pi)^*)U_\gamma(\pi) \cdot \left[\gamma_{I_{\xi_2}^c}(U_\lambda(-\pi)U_0(-\pi)^*)U_\gamma(-\pi) \right]^* \\
&= \gamma_{I_{\xi_1}^c}(\Gamma_\lambda(\pi)^*)U_\gamma(2\pi) \cdot \gamma_{I_{\xi_1}^c}U(\pi)U_\lambda(\pi) \\
&= \gamma_{I_{\xi_1}^c}(\Gamma_\lambda(\pi)^*)S_\gamma \cdot S_\lambda \cdot \gamma_{I_{\xi_2}^c}(\Gamma_\lambda(\pi)).
\end{aligned}$$

So we have:

$$\frac{S_\delta}{S_\gamma S_\delta} = T_e^* \cdot \gamma_{I_{\xi_1}^c}(\Gamma_\lambda(\pi)^*)\gamma_{I_{\xi_2}^c}(\Gamma_\lambda(\pi))T_e.$$

It is clear, by Lemma 1.4.1, that as long as $g.I_0 \cap I_0 = \emptyset$,

$$\gamma_{I_{\xi_1}^c}(\Gamma_\lambda(g)^*)\gamma_{I_{\xi_2}^c}(\Gamma_\lambda(g)) = \gamma_{I_{\xi_1}^c}(\Gamma_\lambda(\pi)^*)\gamma_{I_{\xi_2}^c}(\Gamma_\lambda(\pi)).$$

The proof of the proposition is now completed. \square

An analogue of Proposition 1.4.3 in a special case is proved in §3.3.

Proposition 1.4.3 has an interesting implication. Suppose

$$\sigma_{\gamma\lambda}\sigma_{\lambda\gamma} = \gamma_{I_{\xi_1}^c}(\Gamma_\lambda(g)^*)\gamma_{I_{\xi_2}^c}(\Gamma_\lambda(g))$$

is not identity, i.e. if we can find $\delta \prec \gamma\lambda$ such that $\frac{S_\delta}{S_\lambda S_\gamma} \neq 1$, then it follows that $\Gamma_\lambda(g) \notin \mathcal{A}(J_1) \vee \cdots \vee \mathcal{A}(J_n)$ for any $J_i \subset I_{\xi_1}^c \cap I_{\xi_2}^c, i = 1, \dots, n$.

In fact, by isotony we have

$$\gamma_{I_{\xi_1}^c}(x) = \gamma_{I_{\xi_2}^c}(x) \quad \text{for any } x \in A_J \subset A(I_{\xi_1}^c) \cap A(I_{\xi_2}^c).$$

So if $\Gamma_\lambda(g) \in A(J_1) \vee \cdots \vee A(J_n)$ with $J_i \subset I_{\xi_1}^c \cap I_{\xi_2}^c, i = 1, \dots, n$, then $\sigma_{\lambda\gamma}\sigma_{\gamma\lambda} = \gamma_{I_{\xi_1}^c}(\Gamma_\lambda(g)^*)\gamma_{I_{\xi_2}^c}(\Gamma_\lambda(g)) = \text{id}$, contradicting our assumption that $\sigma_{\gamma\lambda}\sigma_{\lambda\gamma} \neq \text{id}$.

Now suppose we divide the circle into four equal parts and label the segments anti-clock wise by I_0, I_1, I_2, I_3 . Let \tilde{a} be the anti-clockwise rotation by $\frac{\pi}{2}$. Choose $I_1 = I_{\xi_2}, I_3 = I_{\xi_1}$. Notice $\Gamma_\lambda(\tilde{a}^2)^* \in A(I_1^c) \cap A(I_3^c) \supset A(I_0) \vee A(I_2)$. Thus if $\sigma_{\gamma,\lambda} \cdot \sigma_{\lambda,\gamma} \neq 1$, then $\Gamma_\lambda(\tilde{a}^2)^* \notin A(I_0) \vee A(I_1)$.

In §3, we shall see indeed that $\sigma_{\gamma,\lambda} \cdot \sigma_{\lambda,\gamma} \neq 1$ for an interesting class of conformal precosheaves, and we will determine the index and the dual principle graph of the inclusion

$$A_{I_0} \vee A_{I_2} \subset A_{I_1^c} \cap A_{I_3^c}.$$

For another application of Proposition 1.4.3, see Lemma 3.2 of [X].

§2. POSITIVE ENERGY REPRESENTATIONS OF LOOP GROUP

2.1. Basic representation of LU_n .

Let H denote the Hilbert space $L^2(S^1; \mathbb{C}^n)$ of square-summable \mathbb{C}^n -valued functions on the circle. The group LU_n of smooth maps $S^1 \rightarrow U_n$ acts on H multiplication operators.

Let us decompose $H = H_+ \oplus H_-$, where

$$H_+ = \{\text{functions whose negative Fourier coefficients vanish}\}.$$

We denote by P the projection from H onto H_+ .

Denote by $U_{\text{res}}(H)$ the group consisting of unitary operator A on H such that the commutator $[P, A]$ is a Hilbert-Schmidt operator. Denote by $\text{Diff}^+(S^1)$ the group of orientation preserving diffeomorphism of the circle. It follows from Proposition 6.3.1 and Proposition 6.8.2 in [PS] that LU_n and $\text{Diff}^+(S^1)$ are subgroups of $U_{\text{res}}(H)$. There exists a central extension U_{res}^\sim of $U_{\text{res}}(H)$ as defined in §6.6 of [PS].

The central extension $\mathcal{L}U_n$ of LU_n induced by U_{res}^\sim is called the basic extension. We shall denote by \mathcal{D} the induced central extension of $\text{Diff}^+(S^1)$ from U_{res}^\sim .

Let T be the center of $U(n)$. T is isomorphic to S^1 , but we introduce this symbol T since there are many different circles in the theory. We shall denote by $\mathcal{L}T$ the central extension of LT induced from $\mathcal{L}U(n)$.

The basic representation of $\mathcal{L}U_n$ is the representation on Fermionic Fock space $F_p = \Lambda(PH) \otimes \Lambda((1-p)H)^*$ as defined in §10.6 of [PS]. We shall review what we will use in §3. For more details, see [PS] or [W2].

Let $I = \bigcup_{i=1}^n I_i$ be a proper subset of S^1 , where I_i are intervals of S^1 . Denote by $M(I)$ the von Neumann algebra generated by $c(\xi)'s$, with $\xi \in L^2(I, \mathbb{C}^n)$. Here $c(\xi) = a(\xi) + a(\xi)^*$ and $a(\xi)$ is the creation operator defined as in Chapter 1 of [W2]. Let $k : F_\rho \rightarrow F_\rho$ be the Klein transformation given by multiplication by 1 on even forms and by i on odd forms. The proof of the following proposition may be found in §15 of chapter 2 of [W2].

Proposition 2.1.1 (Araki duality).

- (1) *The vacuum vector Ω is cyclic and separating for $M(I)$;*
- (2) $M(I)' = k^{-1}M(I^c)k$.

The irreducible level n representations of $\mathcal{L}T$ are completely classified in chapter 9 of [PS]. There are n such irreducible representations, and we denote the representation by π_i , $i = 0, 1, \dots, n-1$.

We denote by F_i , $i = 0, 1, 2, \dots, n-1$ the corresponding representation space.

The group $(\mathcal{L}T)^\circ$ can be written $T \times V$, where V is a vector space. The extension $(\mathcal{L}T)^\circ$ is $T \times \tilde{V}$. Under the action of $(\mathcal{L}T)^\circ$, F_i decomposes as:

$$F_i = \bigoplus_{d \equiv i \pmod n} F_{i,d}$$

where each $F_{i,d}$ is an irreducible representation of $(\mathcal{L}T)^\circ$ of level n in which the constants $T \subset \mathcal{L}T$ act by $u \rightarrow u^{-d}$. The loops of winding number n in $\mathcal{L}T$ maps $F_{i,d}$ to $F_{i,d+n}$. The action of the central extension of $\text{Diff}^+(S^1)$ preserves each $F_{i,d}$.

Each F_i is a positive energy representation and the lowest energy state of Ω_i has the property that $L_0\Omega_i = \frac{i^2}{2n}\Omega_i$, where L_0 is the generator of the action of the rotation circle group on F_i (see §9.4 of [PS]).

Let $T \times T$ be a subset of \mathcal{LT} (Remember \mathcal{LT} is considered as a subset of \mathcal{LU}_n), where the first T is the kernel of the extension and the second is the canonical copy of T in \mathcal{LT} . It is proved on Page 57 of [PS] that conjugation by a loop of winding number k transforms $T \times T$ by $(u, v) \rightarrow (uv^{-k}, v)$.

We shall use α to denote a map from S^1 to $U(n)$ such that

$$\alpha(e^{i\theta}) = \begin{pmatrix} e^{ig(\theta)} & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix}$$

where there are $n - 1$ 1's on the diagonal, one $e^{ig(\theta)}$ at the $(1, 1)$ entry, and zero elsewhere. We also require $\frac{1}{2\pi}(g(2\pi) - g(0)) = 1$, i.e., the winding number of α is 1. The conjugation by α on \mathcal{LT} lifts uniquely to an action $Ad_{\tilde{\alpha}}$ on \mathcal{LT} . Let us consider the representation $\pi_i \cdot Ad_{\tilde{\alpha}}$ of \mathcal{LT} . By using the fact that $Ad_{\tilde{\alpha}}$ transforms $T \times T$ by $(u, v) \rightarrow (uv^{-1}, v)$ it is easy to see that $\pi_i \cdot Ad_{\tilde{\alpha}} \cong \pi_{i+1}$.

We shall also use a result concerning the decomposition of F_p under the action of $\mathcal{LT} \times \mathcal{LSU}_n$. Here \mathcal{LSU}_n is a subgroup of \mathcal{LU}_n , and T is the center of U_n . It is proved on Page 212 of [PS], and we shall record this result in the following.

Proposition 2.1.2. *Under the action of $\mathcal{LT} \times \mathcal{LSU}_n$ the basic representation F_ρ breaks up into n pieces*

$$F_d \otimes K_d,$$

where F_d are the irreducible representations of $\mathcal{L}\pi$ of level n and K_d are the irreducible representations of \mathcal{LSU}_n of level 1. Here d is well defined modulo n .

Denote by π the representation of \mathcal{LT} on F_p . By Araki duality, we have

$$\pi(x)\pi(y) = (-1)^{w(x)w(y)}\pi(y)\pi(x)$$

for any $x \in \mathcal{L}_I\pi$, $y \in \mathcal{L}_{I^c}\pi$, and $w(x)$, $w(y)$ denote the winding number of x, y respectively. It follows from Proposition 2.1.1 that $\pi_i(x)\pi_i(y) = (-1)^{w(x)w(y)}\pi_i(y)\pi_i(x)$.

We can obtain a local structure on $\pi_i(\mathcal{LT})$ when n is even. In this case all the loops in \mathcal{LT} have even winding numbers. So $\pi_i(\mathcal{L}_IT)$ and $\pi_i(\mathcal{L}_{I^c}T)$ commutes when n is even.

Recall $\alpha : S^1 \rightarrow U(n)$ is defined by

$$\alpha(e^{i\theta}) = \begin{pmatrix} e^{ig(\theta)} & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix}$$

Suppose α is localized on I , i.e. $\alpha \equiv \text{id}$ on I^c . We claim that $\pi_g(Ad_{\tilde{\alpha}^i}x) = (-1)^{i \cdot w(x)}\pi_g(x)$ for any $x \in \mathcal{L}_{I^c}T$ of winding number $w(x)$. In fact we have

$$\pi(Ad_{\tilde{\alpha}^i}x) = (-1)^{i \cdot w(x)}\pi(x)$$

, and it follows that $\pi_j(Ad_{\tilde{\alpha}^i}x) = (-1)^i \pi_j(x)$ since $Ad_{\tilde{\alpha}^i} \cdot x \in \mathcal{LT}$.

Finally let us note if we denote by $\beta : S^1 \rightarrow U(n)$, a map defined by

$$\beta(e^{i\theta}) = e^{ig(\theta)} \cdot \text{id}$$

then $\pi(Ad_{\tilde{\alpha}^n} \cdot x) = \pi(Ad_{\tilde{\beta}} \cdot x)$ since $\alpha^n \beta^{-1} \in LSU(n)$ and \mathcal{LT} commutes with $LSU(n)$.

It follows that

$$\pi_i(Ad_{\tilde{\alpha}^n} \cdot x) = \pi_i(\tilde{\beta})\pi_i(x)\pi_i(\tilde{\beta}^{-1}) \quad \text{for any}$$

$x \in \mathcal{LT}$ since $\tilde{\beta} \in \mathcal{LT}$.

§2.2. Conformal precosheaf from representation of Loop groups.

Let $G = SU(n)$. We denote LG the group of smooth maps $f : S^1 \mapsto G$ under pointwise multiplication. The diffeomorphism group of the circle $\text{Diff}S^1$ is naturally a subgroup of $\text{Aut}(LG)$ with the action given by reparametrization. In particular the group of rotations $\text{Rot}S^1 \simeq U(1)$ acts on LG . We will be interested in the projective unitary representation $\pi : LG \rightarrow U(H)$ that are both irreducible and have positive energy. This means that π should extend to $LG \ltimes \text{Rot} S^1$ so that $H = \oplus_{n \geq 0} H(n)$, where the $H(n)$ are the eigenspace for the action of $\text{Rot}S^1$, i.e., $r_\theta \xi = \exp^{in\theta} \xi$ for $\theta \in H(n)$ and $\dim H(n) < \infty$ with $H(0) \neq 0$. It follows from [PS] that for fixed level m which is a positive integer, there are only finite number of such irreducible representations indexed by the finite set

$$\ddot{P}_+^m = \left\{ \lambda \in P \mid \lambda = \sum_{i=1, \dots, n-1} \lambda_i \Lambda_i, \lambda_i \geq 0, \sum_{i=1, \dots, n-1} \lambda_i \leq m \right\}$$

where P is the weight lattice of $SU(n)$ and Λ_i are the fundamental representations. We will use Λ_0 to denote the trivial representation of $SU(n)$. For $\lambda, \mu, \nu \in \ddot{P}_+^m$, define $N_{\lambda\mu}^\nu = \sum_{\delta \in \ddot{P}_+^m} S_\lambda^\delta S_\mu^\delta S_\nu^{\delta*} / S_{\Lambda_0}^\delta$ where S_λ^δ is given by the Kac-Peterson formula:

$$S_\lambda^\delta = c \sum_{w \in S_n} \varepsilon_w \exp(iw(\delta) \cdot \lambda 2\pi/n)$$

where $\varepsilon_w = \det(w)$ and c is a normalization constant fixed by the requirement that S_μ^δ is an orthonormal system. It is shown in [K2] that $N_{\lambda\mu}^\nu$ are non-negative integers. Moreover, define $\ddot{G}r_m$ to be the ring whose basis are elements of \ddot{P}_+^m with structure constants $N_{\lambda\mu}^\nu$. The natural involution $*$ on \ddot{P}_+^m is defined by $\lambda \mapsto \lambda^* =$ the conjugate of λ as representation of $SU(n)$.

We shall also denote $S_{\Lambda_0}^\lambda$ by $S(\lambda)$. Define $d_\lambda = \frac{S(\lambda)}{S(\Lambda_0)}$. We shall call (S_ν^ρ) the S -matrix of $LSU(n)$.

It follows from [K2] that S -matrix is symmetric and unitary. In particular, $\sum_{\lambda \in \check{P}_+^m} d_\lambda^2 = \frac{1}{S(\Lambda_0)^2}$.

The irreducible positive energy representations of $LSU(n)$ at level m give rise to an irreducible conformal precosheaf \mathcal{A} (see §2) and its covariant representations in the following way:

First note if π_λ is a representation of central extension of $\text{Diff}^+(S^1)$ on H_λ , then π_λ induces an action of \mathbf{G} in the following way:

Let us denote the induced central extension of $PSL(2, R) \subset \text{Diff}^+(S^1)$ from that of $\text{Diff}^+(S^1)$ by $\widetilde{PSL}(2, R)$. Let $\pi_2 : \widetilde{PSL}(2, R) \rightarrow PSL(2, R)$ and $\pi_1 : \mathbf{G} \rightarrow PSL(2, R)$ be the natural covering maps. Since \mathbf{G} is simply connected, there exists a homomorphism φ from \mathbf{G} to $\widetilde{PSL}(2, R)$ such that $\pi_2 \cdot \varphi = \pi_1$. We shall fix φ and denote by $\pi_\lambda(g)$ the operator $\pi_\lambda(\varphi(g))$ for any $g \in \mathbf{G}$ in the following.

The conformal precosheaf is defined by

$$\mathcal{A}(I) = \pi_0(\mathcal{L}_I G)''.$$

In fact, by the results in Chapter 2 of [W2] that $\mathcal{A}(I)$ satisfies A to F of §1.2 and therefore is indeed an irreducible conformal precosheaf.

Let $U(\lambda, I)$ be a unitary operator from H_λ to H_0 such that:

$$\pi_\lambda(x) = U(\lambda, I)^* \pi_0(x) U(\lambda, I)$$

for any $x \in \mathcal{L}_I G$.

Fix $I_1 \subset S^1$. We define a collection of maps as follows. For any interval $J \subset S^1$, $x \in \mathcal{A}(J)$,

$$\lambda_J(x) = U(\lambda, I_1^c) U^*(\lambda, J) x U(\lambda, J) U(\lambda, I_1^c)^*.$$

It follows that if $J \supset I_1$, then $\lambda_J(x)$ commutes with $\mathcal{A}(J^c)$ for any $x \in \mathcal{A}(J)$. By Haag-duality, if $J \supset I_1$, $\lambda_J(\mathcal{A}_J) \subset \mathcal{A}(J)$.

Define:

$$U_\lambda(g) = U(\lambda, I^c) \pi_\lambda(g) U(\lambda, I^c)^*.$$

It is easy to check that $\{\lambda_J\}$ gives a covariant representation of conformal precosheaf \mathcal{A} . Let us note that the intervals in §1.2 are defined to be open intervals. We can actually choose the interval to be closed since we shall be concerning with the conformal precosheaves from positive energy representations of LG and by Theorem E of [W2], $\pi(L_I G)'' = \pi(L_{\bar{I}} G)''$ where \bar{I} is the closure of I .

The collection of maps $\{\lambda_J\}$ define an endomorphism λ of $C^*(\mathcal{A})$ (See [GL2], Section 8). The relation between λ and λ_J is given by

$$\pi_0(\lambda(i_J(x))) = \lambda_J(x) \quad \text{for any } x \in \mathcal{A}(J).$$

Here $i_J : \mathcal{A}(J) \rightarrow C^*(\mathcal{A})$ is the embedding of $\mathcal{A}(J)$ in $C^*(\mathcal{A})$, and π_0 is the vacuum representation of $C^*(\mathcal{A})$.

This makes the definition of composition $\lambda \cdot \mu$ of two covariant representations $\{\lambda_J\}$ and $\{\mu_J\}$ straightforward. One simply define $\lambda \cdot \mu$ as the composition of λ, μ as endomorphisms of $C^*(\mathcal{A})$. It is easy to check that if $J \supset I_1$

$$\pi_0((\lambda \circ \mu)(i_J(x))) = \lambda_J \circ \mu_J(x).$$

An equivalent definition can be found in [Fro].

The following remarkable result is proved in [W2] (See Corollary 1 of Chapter V in [W2]).

Theorem 2.2. *Each $\lambda \in \ddot{P}_+^m$ has finite index with index value d_λ^2 . The fusion ring generated by all $\lambda \in \ddot{P}_+^m$ is isomorphic to $\ddot{G}r_m$.*

The equivalence between the ring structure described in Corollary 1 of Chapter V in [W2] and $\ddot{G}r_m$ described above is proved on Page 288 of [K2].

Similarly, the positive energy representations of LT at even level n give rise to an irreducible conformal precosheaf \mathcal{A} and its covariant representations: We simply take $\mathcal{A}(I) = \pi_0(\mathcal{L}_I T)''$. Notice the locality in this case follows from the end of §2.1. Recall from §2.1

$$\alpha(e^{i\theta}) = \begin{pmatrix} e^{ig(\theta)} & & \\ & 1 & \\ & & \ddots & 1 \end{pmatrix}$$

If we choose $g(\theta) \equiv 0$ on I^c , then the adjoint action $Ad_{\tilde{\alpha}}$ of $\tilde{\alpha}$ on $\mathcal{L}T$ gives rise to a localized automorphism, which we shall denote by the same notation $Ad_{\tilde{\alpha}}$, of $\mathcal{A}(I)$. By our choice, $Ad_{\tilde{\alpha}}$ is localized on I . Moreover,

$$\pi_i \simeq \pi_0 \circ Ad_{\tilde{\alpha}}^i.$$

From the end of §2.1, the adjoint action of $\tilde{\alpha}^n$ on $\mathcal{A}(I)$ is inner. It follows that the fusion ring generated by $\pi_i \quad i = 0, 1, \dots, n-1$ of \mathcal{A} is isomorphic to the group ring of \mathbb{Z}_n .

In the case n is odd, $\mathcal{A}(I)$ does not satisfy locality conditions. (See the end of §2.1) We shall deal with this case in §3.3.

2.3. Local Factorization.

We shall use the local factorization properties for free fermions and $LSU(N)$ in §3. These results are well known, see e.g., [B], [W3]. Let us first prepare some notations. Fix $I = I_1 \cup I_3$, $\bar{I}_1 \cap \bar{I}_3 = \emptyset$, $\bar{I} \subsetneq S^1$. We assume I_1 is an interval of S^1 , and I_2 is a finite union of intervals of S^1 . For a bounded operator $A : F_p \rightarrow F_p$, we define $A^+ = \Gamma A \Gamma$, $A^- = A - A^+$, where Γ is an operator on F_p given by multiplication by 1 on even form stand -1 on odd forms. An operator A is called even (resp. odd) if $A = A^+$ (resp. $A = A^-$). For any algebra $M \subset B(F_p)$ we denote by M^e the subalgebra of M consisting of even operators in M .

Lemma 2.3.1. (1) $M(I_1)^e \subset M(I_1)$ is an inclusion of subfactors with index 2 and $M(I_1) \subset \{M(I_1), \Gamma\}''$ is a basic construction for $M(I_1)^e \subset M(I_1)$.
 (2) $M(I_1) = \pi(L_{I_1}U(n))''$.

Proof: (1) By Takesaki devisage as in [W2], $M(I_1)^e$ is a factor. We noticed that there is a \mathbb{Z}_2 -action on $M(I_1)$ given by: $\alpha(x) = \Gamma x \Gamma$ for any $x \in M(I_1)$, and $M(Z_1)^e$ is the fixed point algebra. Again by Takesaki's devisage as in [W2], $M(I_1) \subset \{M(I_1), \Gamma\}''$ is the basic construction for $M(I_1)^e \subset M(I_1)$, and $M(I_1)^e = M(I_1)$ iff $M(I_1)^e \Omega = M(I_1) \Omega$. But $M(I_1)^e \Omega \subsetneq M(I_1) \Omega$, it follows that such an action is properly outer and the index of $M(I_1)^e \subset M(I_1)$ is 2.

(2) We just have to show that if I is a connected interval, then $\pi(L_I U(n))'' = M(I)$. By §12 of [W3], we just need to show $\pi(L_I U(n))\Omega = \overline{M(I)\Omega}$, where Ω is the vacuum vector in F_p . But Ω is both separating and cyclic for $\pi(L_I U(n))''$ and $M(I)$, it follows that $\pi(L_I U(n))'' = M(I)$. \square

We define a graded tensor product \otimes_2 by the following formula:

$$A \otimes_2 B = A \otimes B^+ + A\Gamma \otimes B^-$$

$A \otimes_2 B$ is considered as an operator on Hilbert space tensor product $F_p \otimes F_p$.

Let A_1, A_2, B_1, B_2 be even or odd operators, i.e. $\Gamma A_i \Gamma = A_i$ or $-A_i$, $\Gamma B_i \Gamma = B_i$ or $-B_i$, $i = 1, 2$. Define $d(A) = 0$ or 1 if $\Gamma A_i \Gamma = A$ or $-A$.

It follows from the definition of \otimes_2 that:

$$\begin{aligned} (A_1 \otimes_2 B_1)^* &= (-1)^{d(A_1)d(B_1)} A_1^* \otimes B_1^* \\ (A_1 \otimes_2 B_1) \cdot (A_2 \otimes_2 B_2) &= (-1)^{d(B_1)d(A_2)} A_1 A_2 \otimes_2 B_1 B_2. \end{aligned}$$

For $A \in M(I_1)$, $B \in M(I_3)$, we define

$$\varphi_1(A \otimes_2 B) = \langle AB\Omega, \Omega \rangle$$

where Ω is the vacuum vector in F_p .

Let \mathcal{H}_π be an irreducible positive energy representation of $LSU(m)$. We shall denote by $N(I_1)$ (resp. $N(I_3)$) the von Neumann algebra $\pi(L_{I_1}SU(m))''$ (resp. $\pi(L_{I_3}SU(m))''$).

For $A \in N(I_1)$, $B \in N(I_3)$, we define

$$\varphi_2(A \otimes B) = \langle AB\Omega, \Omega \rangle.$$

Proposition 2.3.1. (1) φ_2 extends to a normal faithful state on $N(I_1) \hat{\otimes} N(I_3)$ where $\hat{\otimes}$ is von-Neumann algebra tensor product. There exists a unitary operator $U_2 : H_\pi \rightarrow H_\pi \otimes H_\pi$ such that $U_2 A B U_2^* = A \otimes B$ for any $A \in N(I_1)$, $B \in N(I_3)$. U_2 implements a spatial isomorphism between

$$(N(I_1) \vee N(I_3))'' \quad \text{and} \quad N(I_1) \hat{\otimes} N(I_3).$$

(2) φ_1 extends to a normal faithful state on von Neumann algebra $\{A \otimes_2 B, A \in M(I_1), B \in M(I_2)\}''$ (denoted by $M(I_1) \hat{\otimes}_2 M(I_2)$) on $F_p \otimes F_p$. There exists a unitary operator $U_1 : F_p \rightarrow F_p \otimes F_p$ such that:

$$U_1 A B U_1^* = A \otimes_2 B \quad \text{for any} \quad A \in M(I_1), B \in M(I_2).$$

Proof: (1) The proof is given here as in [W3] (also see [B]). Since we have a positive energy representation $U(z) = z^{L_0}$ of the circle group on \mathcal{H} with vacuum vector Ω and that x and y lie in disjoint local algebras $N(I_1)$ and $N(I_2)$, we have $[x, U(z)yU(z)^{-1}] = 0$ for z near 1, say on an arc I with end points a_{\pm} with $\bar{a}_+ = a_-$. Define $f_+(z) = (xz^{L_0}y\Omega, \Omega)$ for $|z| \leq 1$ and $f_-(z) = (yz^{-L_0}x\Omega, \Omega)$ for $|z| \geq 1$. The commutativity condition on x and y shows that f_+ and f_- agree on I so jointly define a holomorphic function f on $\mathbb{C} \setminus I^c$. Let $g(z) = \exp(-\alpha(z/a - 1)^{-3/2} - \bar{\alpha}(z/\bar{a} - 1)^{-3/2})$ for z in $\mathbb{C} \setminus I^c \cup (-\infty, -1]$, where $\alpha = \exp(-i\pi/4)$. This holomorphic function blows up at a_{\pm} ; however in the closed sector S bounded by the radii through a_{\pm} it satisfies $|g(z)| \leq 1$ and is continuous. Let Γ be any simple closed contour in S , coinciding with the radii near a_{\pm} and winding round 1 once. If D is the domain enclosed by Γ , fg is holomorphic on D and continuous on \bar{D} . By Cauchy's theorem $2\pi i f(1)g(1) = \int_{\Gamma} g(z)f(z)(z-1)^{-1}dz$. Let Γ_+, Γ_- be the parts of the contour inside and outside the unit disc. Because $|g(ra_{\pm})| \sim \exp(-(2|r-1|)^{-3/2})$ and there is an asymptotic estimate (cf.[K1]) $\text{Tr}(|(ra_{\pm})^{L_0}|) = \text{Tr}(|r|^{L_0}) \sim \exp(-C/\log r)$ with $C > 0$ as $r \uparrow 1$, we see that

$$A_{\pm} = \frac{1}{2\pi i g(1)} \int_{\Gamma_{\pm}} g(z)f(z)z^{\pm L_0}(z-1)^{-1}dz$$

are trace class operators such that

$$(xy\Omega, \Omega) = f(1) = (xA_+y\Omega, \Omega) + (yA_-x\Omega, \Omega)$$

for $x \in N(I_1)$ and $y \in N(I_2)$. Since A_+ and A_- are trace class, the right hand side extends to a normal form on $N(I_1) \otimes M(I_2)$ which is a state ω in view of the form of the left hand side. The representation π_{ω} of $N(I_1) \otimes N(I_2)$ is faithful (since the algebra is a factor) and may be canonically identified with the obvious representation on the closure of $N(I_1)N(I_2)\Omega$. By the Roth-Schlieder theorem as in [W2] this is dense and thus π_{ω} gives an isomorphism of $N(I_1) \otimes N(I_2)$ onto the von Neumann algebra generated by $N(I_1)$ and $N(I_2)$. Because everything is type III, this isomorphism can be implemented by a unitary.

(2) The proof is essentially the same as in (1) with some necessary modifications. Let us write $\varphi_1(A \otimes_2 B)$ as follows:

$$\begin{aligned} \varphi_1(A \otimes_2 B) &= \langle AB\Omega, \Omega \rangle \\ &= \langle A^+B^+\Omega, \Omega \rangle + \langle A^+B^-\Omega, \Omega \rangle + \langle A^-B^+\Omega, \Omega \rangle + \langle A^-B^-\Omega, \Omega \rangle. \end{aligned}$$

Since $[A^+, U(z)B^\pm U(z)^{-1}] = 0$, $[A^-, U(z)B^+ U(z)^{-1}] = 0$ for z close to 1, the same argument as in (1) shows

$$\begin{aligned}\langle A^+ B^+ \Omega, \Omega \rangle &= \sum_i a_i^{(++)} \langle A^+ \otimes B^+ \xi_i^{(++)}, \eta_i^{(++)} \rangle \\ \langle A^+ B^- \Omega, \Omega \rangle &= \sum_i a_i^{(+-)} \langle A^+ \otimes B^- \xi_i^{(+-)}, \eta_i^{(+-)} \rangle \\ \langle A^- B^+ \Omega, \Omega \rangle &= \sum_i a_i^{(-+)} \langle A^- \otimes B^+ \xi_i^{(-+)}, \eta_i^{(-+)} \rangle\end{aligned}$$

where $\{\xi_i^{(++)}\}$ (resp. $\{\eta_i^{(++)}\}$, $\{\xi_i^{(+-)}\}$, $\{\eta_i^{(+-)}\}$, $\{\xi_i^{(-+)}\}$, $\{\eta_i^{(-+)}\}$) are orthonormal basis in $H \otimes H$ and

$$\sum_i |a_i^{(++)}| < \infty, \quad \sum_i |a_i^{(+-)}| < \infty, \quad \sum_i |a_i^{(-+)}| < \infty.$$

As for $\langle A^- B^- \Omega, \Omega \rangle$, since we have

$$[A^-, U(z)B^- U(z)^{-1}]_- = 0,$$

essentially the same argument as in (1) shows

$$\langle A^- B^- \Omega, \Omega \rangle = \sum_i a_i^{(--) } \langle A^- \otimes B^- \xi_i^{(--)}, \eta_i^{(--) } \rangle$$

with $\{\xi_i^{(--)}\}$ (resp. $\{\eta_i^{(--)}\}$) orthonormal basis of $F_p \otimes F_p$ and $\sum_i |a_i^{(--) }| < \infty$.

Notice

$$\begin{aligned}A \otimes B^+ &= \frac{1}{2}[(A \otimes_2 B) + (1 \otimes \Gamma)(A \otimes_2 B)(1 \otimes \Gamma)], \\ A\Gamma \otimes B^- &= \frac{1}{2}[A \otimes_2 B - (1 \otimes \Gamma)(A \otimes_2 B)(1 \otimes \Gamma)] \\ \langle A^+ B^- \Omega, \Omega \rangle &= \langle \Gamma A^+ B^- \Omega, \Omega \rangle \\ \langle A^- B^- \Omega, \Omega \rangle &= \langle \Gamma A^- B^- \Omega, \Omega \rangle \\ \Gamma A^+ \otimes B^- &= \frac{1}{2}[(\Gamma \otimes 1)(A\Gamma \otimes B^-)(\Gamma \otimes 1) + (A\Gamma \otimes B^-)] \\ \Gamma A^- \otimes B^- &= \frac{1}{2}[-(\Gamma \otimes 1)(A\Gamma \otimes B^-)(\Gamma \otimes 1) + A\Gamma \otimes B^-].\end{aligned}$$

It follows that

$$\varphi_1(A \otimes_2 B) = \sum_{i=1}^8 \psi_i(A \otimes_2 B)$$

where each $\psi_i(A \otimes_2 B) = \sum_{j=1}^\infty b_{ij} \langle (A \otimes_2 B) \xi_{i,j}, \eta_{i,j} \rangle$ with $\sum_{j=1}^\infty |b_{i,j}| < \infty$ and $\{\xi_{i,j}\}$ (resp. $\{\eta_{i,j}\}$) orthonormal basis in $F_p \otimes F_p$.

Hence φ_1 extends to a normal state on the von Neumann algebra generated by $M = \{A \otimes_2 B, A \in M(I_1), B \in M(I_3)\}''$ on $F_p \otimes F_p$. Let us show M is a hyperfinite III_1 factor. Denote by $\tilde{M} = \{M(I_1), \Gamma\}'' \hat{\otimes} M(I_3)$. We have $M \subset \tilde{M} = \{M, \Gamma \otimes 1\}''$.

We claim $M'' \vee \tilde{M}' = B(F_p \otimes F_p)$. In fact, since $M(I_1) \otimes 1$ and $\{M(I_1), \Gamma\}' \otimes 1$ are in $M'' \vee \tilde{M}'$ and $M(I_1) \vee \{M(I_1), \Gamma\}' = B(F_p)$, by Lemma 2.3.1. It follows that $1 \otimes M(I_3)$ and $1 \otimes M(I_3)'$ are in $M'' \vee \tilde{M}'$.

Hence $M'' \vee \tilde{M}' = B(F_p \otimes H_p)$ and we have $M' \cap \tilde{M} = \mathbb{C}1$. This shows that M is a factor. Since $\Gamma \otimes 1 \cdot M \cdot \Gamma \otimes 1 \subset M$, $\tilde{M} = M \rtimes \mathbb{Z}_2$, where the action of \mathbb{Z}_2 on M is given by conjugation of $\Gamma \otimes 1$. If this action is inner, since both M and \tilde{M} are factors, we must have $\Gamma \otimes 1 \in M$. So $M = \tilde{M}$. It follows that $M(I_1) \otimes M(I_2) = \tilde{M}(I_1) \otimes M(I_2)$. But $M(I_1) \otimes M(I_2) \subset \tilde{M}(I_1) \otimes M(I_2)$ has index 2 by Lemma 2.3.1, this is a contradiction. So the action is outer, and hence properly out by the factoriality of M and \tilde{M} .

It follows that $M \subset \tilde{M}$ has index 2. Since \tilde{M} is a hyperfinite III_1 factor, so is M . Recall φ_1 is a normal state on the hyperfinite III_1 factor M . φ_1 must be faithful. The GNS representation π_{φ_1} of M is faithful and can be canonically identified with the obvious representation on the closure of $M(I_1)M(I_3)\Omega$.

By the Roth-Schlieder theorem this is F_p and thus π_{φ_1} gives an isomorphism of $M(I_1) \vee M(I_3)$ onto the von Neumann algebra M . Because M is a type III_1 factor, this isomorphism can be implemented by a unitary $U_1 : F_p \rightarrow F_p \otimes F_p$ such that $U_1 A B U_1^* = A \otimes_2 B$ for any $A \in M(I_1)$, $B \in M(I_3)$. \square

§3. JONES-WASSERMANN SUBFACTORS FOR DISJOINT INTERVALS

3.1. Conformal inclusions.

Let $H \subset G$ be inclusions of compact Lie groups. $H \subset G$ is called a conformal inclusion if every level 1 irreducible projective positive energy representations of LG decomposes as a finite number of irreducible projective representations of LH . A list of conformal inclusions can be found in [GNO].

We shall be interested in the following two conformal inclusions:

$$\begin{aligned} L(SU(m) \times SU(n)) &\subset L SU(nm) \\ LU(1) \times LSU(n) &\subset LU(n). \end{aligned}$$

Let π^0 be the vacuum representation of $LSU(nm)$ on Hilbert space H^0 . The decomposition of π^0 under $L(SU(m) \times SU(n))$ is known, see, e.g. [Itz]. To describe such a decomposition, let us prepare some notation. We shall use \dot{S} to denote the S -matrices of $SU(m)$, (see §2.2), and \ddot{S} to denote the S -matrices of $SU(n)$. The level n (resp. m) weight of $LSU(m)$ (resp. $LSU(n)$) will be denoted by $\dot{\lambda}$ (resp. $\ddot{\lambda}$).

Only in this section we will use $\ddot{\lambda}$ to denote the weights of $LSU(n)$ to distinguish them from the weights of $LSU(m)$. We have used λ to denote the the weights of $LSU(n)$ in the rest of this paper where no confusion may arise to simply notations.

We start by describing \dot{P}_+^n (resp. \ddot{P}_+^m), i.e. the highest weights of level n of $LSU(m)$ (resp. level m of $LSU(n)$).

\dot{P}_+^n is the set of weights

$$\dot{\lambda} = \tilde{k}_0 \dot{\Lambda}_0 + \tilde{k}_1 \dot{\Lambda}_1 + \cdots + \tilde{k}_{m-1} \dot{\Lambda}_{m-1}$$

where \tilde{k}_i are non-negative integers such that

$$\sum_{i=0}^{m-1} \tilde{k}_i = n$$

and $\dot{\Lambda}_i = \dot{\Lambda}_0 + \dot{\omega}_i$, $1 \leq i \leq m-1$, where $\dot{\omega}_i$ are the fundamental weights of $SU(m)$.

Instead of $\dot{\lambda}$ it will be more convenient to use

$$\dot{\lambda} + \dot{\rho} = \sum_{i=0}^{m-1} k_i \dot{\Lambda}_i$$

with $k_i = \tilde{k}_i + 1$ and $\sum_{i=0}^{m-1} k_i = m + n$. Due to the cyclic symmetry of the extended Dynkin diagram of $SU(m)$, the group \mathbb{Z}_m acts on \dot{P}_+^n by

$$\dot{\Lambda}_i \rightarrow \dot{\Lambda}_{(i+\sigma) \bmod m}, \quad \sigma \in \mathbb{Z}_m.$$

Let $\Omega_{m,n} = \dot{P}_+^n / \mathbb{Z}_m$. Then there is a natural bijection between $\Omega_{m,n}$ and $\Omega_{n,m}$ (see §2 of [Itz]).

We shall parametrize the bijection by a map

$$\beta : \dot{P}_+^n \rightarrow \ddot{P}_+^m$$

as follows. Set

$$r_j = \sum_{i=j}^m k_i, \quad 1 \leq j \leq m$$

where $k_m \equiv k_0$. The sequence (r_1, \dots, r_m) is decreasing, $m + n = r_1 > r_2 > \cdots > r_m \geq 1$. Take the complementary sequence $(\bar{r}_1, \bar{r}_2, \dots, \bar{r}_n)$ in $\{1, 2, \dots, m+n\}$ with $\bar{r}_1 > \bar{r}_2 > \cdots > \bar{r}_n$. Put

$$S_j = m + n + \bar{r}_n - \bar{r}_{n-j+1}, \quad 1 \leq j \leq n.$$

Then $m + n = s_1 > s_2 > \cdots > s_n \geq 1$. The map β is defined by

$$(r_1, \dots, r_m) \rightarrow (s_1, \dots, s_n).$$

The following lemma summarizes what we will use in §3.4. For the proof, see Lemma 3, 4 of [Itz].

Lemma 3.1.1. (1) Let \dot{Q} be the root lattice of $SU(m)$, $\dot{\Lambda}_i$, $0 \leq i \leq m-1$ its fundamental weights and $\dot{Q}_0 = (\dot{Q} + \dot{\Lambda}_0) \cap \dot{P}_+^n$. Then for each $\dot{\lambda} \in \dot{Q}_0$, there exists a unique $\ddot{\lambda} \in \ddot{P}_+^m$ with $\ddot{\lambda} = \sigma\beta(\dot{\lambda})$ for some $\sigma \in \mathbb{Z}_n$ such that $H_{\dot{\lambda}} \otimes H_{\ddot{\lambda}}$ appears once and only once in H^0 . Moreover, H^0 , as representations of $L(SU(m) \times SU(n))$, is a direct sum of all such $H_{\dot{\lambda}} \otimes H_{\ddot{\lambda}}$.

$$(2) \sum_{\dot{\lambda} \in \dot{Q}_0} (\dot{S}(\dot{\lambda}))^2 = \frac{1}{m}.$$

$$(3) \dot{S}(\dot{\lambda}) = \left(\frac{n}{m}\right)^{\frac{1}{2}} \ddot{S}(\sigma\beta(\dot{\lambda})).$$

Let J be a proper interval of S^1 . We claim $\pi^0(L_J(SU(m) \times SU(n)))''$ is a factor. In fact we can show $\pi^0(L_J(SU(m) \times SU(n)))' \cap \pi_0(L_J SU(mn))'' = \mathbb{C}1$. Suppose $a \in \pi^0(L_J(SU(m) \times SU(n)))'' \cap \pi^0(L_J SU(mn))'$, then

$$\begin{aligned} a &\in \pi^0(L_J(SU(m) \times SU(n)))' \cap \pi^0(L_{J^c}(SU(m) \times SU(n)))' \\ &= \pi^0(L(SU(m) \times SU(n)))''. \end{aligned}$$

Hence a is a sum of projections $\rho_{\dot{\lambda}\ddot{\lambda}}$ which maps H^0 to $H_{\dot{\lambda}} \otimes H_{\ddot{\lambda}}$. By (1) of Lemma 3.1.1, the vacuum vector Ω of H^0 appears only once in $H_{\dot{\lambda}_0} \otimes H_{\ddot{\lambda}_0}$ so $a.\Omega = c\Omega$ with $c \in \mathbb{C}$. Since Ω is separating for $\pi^0(L_J SU(mn))''$ and $a \in \pi^0(L_J SU(mn))''$, it follows that a must be c , a scalar.

From the argument above we obtain an inclusion of irreducible subfactors:

$$\pi^0(L_J(SU(m) \times SU(n)))'' \subset \pi^0(L_J SU(mn))''.$$

The statistical dimension d of the above inclusion is independent of J because of the projective action of $\text{Diff}^+ S^1$ as in the proof of Prop.2.1 of [GL1]. Consider the following analogue of basic construction

$$\begin{aligned} \pi^0(L_J(SU(m) \times SU(n)))'' &\subset \pi^0(L_J SU(mn))'' = \\ \pi^0(L_{J^c} SU(mn))' &\subset \pi^0(L_{J^c}(SU(m) \times SU(n)))'. \end{aligned}$$

It follows from Lemma 3.1.1 and Theorem 2.2 that

$$\begin{aligned} d^2(\pi^0(L_J(SU(m) \times SU(n)))'' \subset \pi^0(L_J SU(mn))'') &= \\ d(\pi^0(L_J(SU(m) \times SU(n)))'' \subset \pi^0(L_{J^c}(SU(m) \times SU(n)))') &= \\ = \sum_{\dot{\lambda} \in \dot{Q}_0} \frac{\dot{S}(\dot{\lambda})^2}{\dot{S}(\dot{\Lambda}_0)^2} = \frac{1}{m\dot{S}(\dot{\Lambda}_0)^2} = \frac{1}{n\ddot{S}(\dot{\Lambda}_0)^2}. \end{aligned}$$

If J is a $\ell - 1$ -disconnected interval, by Proposition 2.3, we have

$$\begin{aligned} d^2(\pi^0(L_J(SU(m) \times SU(n)))'' \subset \pi^0(L_J SU(mn))'') &= \\ = \frac{1}{m^\ell \cdot \dot{S}(\dot{\Lambda}_0)^{2\ell}} = \frac{1}{n^\ell \cdot \ddot{S}(\dot{\Lambda}_0)^{2\ell}}. \end{aligned}$$

Simiarly as above and use Proposition 2.1.1 and 2.1.2 , we have if J is connected then:

$$d(\pi^0(L_J(U(1) \times SU(n)))'' \subset \pi^0(L_J U(n))'' = n^{\frac{1}{2}}.$$

We shall prove , by induction on ℓ , that if J is a $\ell - 1$ -disconnected interval, then

$$d(\pi^0(L_J(U(1) \times SU(n)))'' \subset \pi^0(L_J U(n))'' = n^{\frac{\ell}{2}}.$$

If $\ell = 1$, it is already noted above. Suppose the formula is proved for $\ell < k$, let us prove it for $\ell = k$. Let $J = \widetilde{I_1 \cup I_2}$, where I_1 is an interval and I_2 is an $k - 2$ -disconnected interval. Let $\widetilde{M(I_1)} = (M(I_1) \otimes 1, \Gamma \otimes 1)''$, $N(I_1) = (\pi^0(L_{I_1}(U(1) \times SU(n)))''$, $\widetilde{N(I_1)} = (\pi^0(L_{I_1}(U(1) \times SU(n)), \Gamma \otimes 1))''$. Recall $M(I_1) = \pi^0(L_{I_1} U(n))''$. By using (2) of Proposition 2.3.1, we just have to show

$$N(I_1) \hat{\otimes}_2 N(I_2) \subset M(I_1) \hat{\otimes}_2 M(I_2)$$

has index $n^{\frac{k}{2}}$.

We claim $d^2(M(I_1) \hat{\otimes}_2 M(I_2) \subset \widetilde{M(I_1) \hat{\otimes}_2 M(I_2)}) = 2$.

Notice conjugation by $\Gamma \otimes 1$ induces a \mathbb{Z}_2 action on $M(I_1) \otimes M(I_2)$.

Since both $\widetilde{M(I_1) \hat{\otimes}_2 M(I_2)}$ and $M(I_1) \hat{\otimes}_2 M(I_2)$ are type III₁ factors, we just have to show that the conjugate action by $\Gamma \otimes 1$ is not inner. If it is, then

$$\Gamma \otimes 1 \in M(I_1) \hat{\otimes}_2 M(I_2) = \widetilde{M(I_1) \hat{\otimes}_2 M(I_2)}.$$

But from Lemma 2.3.1, we have:

$$d^2(M(I_1) \hat{\otimes}_2 M(I_2) \subset \widetilde{M(I_1) \hat{\otimes}_2 M(I_2)}) = d^2(M(I_1) \subset \widetilde{M(I_1)}) = 2$$

, a contradiction. Thus the conjugate action by $\Gamma \otimes 1$ on $M(I_2)$ is outer, and hence properly outer since both $M(I_1) \otimes M(I_2)$ and $\widetilde{M(I_1) \hat{\otimes}_2 M(I_2)}$ are factors. It follows that

$$d^2(M(I_1) \hat{\otimes}_2 M(I_2) \subset \widetilde{M(I_1) \hat{\otimes}_2 M(I_2)}) = 2.$$

From exactly the same argument as in Lemma 2.3.1 and above we have $d^2(N(I_1) \subset \widetilde{N(I_1)}) = 2$ and

$$d^2(N(I_1) \hat{\otimes}_2 N(I_2) \subset \widetilde{N(I_1) \hat{\otimes}_2 N(I_2)}) = 2.$$

Now by induction hypothesis:

$$d(\widetilde{N(I_1) \hat{\otimes}_2 N(I_2)} \subset \widetilde{M(I_1) \hat{\otimes}_2 M(I_2)}) = d(\widetilde{N(I_1)} \subset \widetilde{M(I_1)}) \cdot n^{\frac{k-1}{2}}.$$

But

$$\begin{aligned} d(\widetilde{N(I_1)} \subset \widetilde{M(I_1)}) &= d(N(I_1) \subset M(I_1)) \cdot d(M(I_1) \subset \widetilde{M(I_1)}) \\ &\quad \cdot d^{-1}(N(I_1) \subset \widetilde{N(I_1)}) \\ &= d(N(I_1) \subset M(I_1)). \end{aligned}$$

$$\begin{aligned}
d(N(I_1) \hat{\otimes}_2 N(I_2) \subset M(I_1) \hat{\otimes}_2 M(I_2)) &\subset d(\widetilde{M(I_1)} \hat{\otimes} M(I_2)) \\
&= d(\widetilde{N(I_1)} \hat{\otimes} N(I_2) \subset \widetilde{M(I_1)} \hat{\otimes} M(I_2)) \\
&\quad \cdot d(N(I_1) \hat{\otimes}_2 N(I_2) \subset \widetilde{N(I_1)} \hat{\otimes}_2 N(I_2)).
\end{aligned}$$

It follows that

$$\begin{aligned}
d(N(I_1) \hat{\otimes}_2 N(I_2) \subset M(I_1) \hat{\otimes}_2 M(I_2)) &= d(\widetilde{N(I_1)} \hat{\otimes} N(I_2) \subset \widetilde{M(I_1)} \hat{\otimes} M(I_2)) \\
&= n^{\frac{k}{2}}.
\end{aligned}$$

By induction hypothesis, we have proved that if J is $\ell - 1$ -disconnected, then $d(N(J) \subset M(J)) = n^{\frac{k}{2}}$.

Let us record what we have proved above in the following proposition.

Proposition 3.1.1. *Suppose J is a $\ell - 1$ -disconnected interval. Then*

- (1) $d^2(\pi^0(L_J(SU(m) \times SU(n)))'' \subset \pi^0(L_J SU(mn)))'' = \frac{1}{n^\ell \cdot \tilde{S}(\dots \Lambda_0)^{2\ell}}$
- (2) $d^2(\pi^0(L_J(U(1) \times SU(n)))'' \subset \pi^0(L_J U(n)))'' = n^\ell$

where in (1), π^0 denotes the level 1 vacuum representation of $\mathcal{L}SU(mn)$ and in (2), π^0 denotes the level 1 vacuum representation of $\mathcal{L}U(n)$.

3.2. Jones-Wasserman Subfactors for Disconnected Intervals.

Fix level $m \geq 1$ and $I = \bigcup_{i=1}^\ell I_i$ is a $\ell - 1$ -disconnected interval. Let $\lambda \in \ddot{P}_m^+$. Then $\pi_\lambda(L_I SU(n))'' \subset \pi_\lambda(L_{I^c} SU(n))'$ is an irreducible inclusion of hyperfinite type III₁ factors.

Since the representation π_λ admits an intertwining projective action of $\text{Diff}^+(S^1)$, it is easy to see that the index of the Jones-Wassermann subfactor depends on I only through the disconnectedness $\ell - 1$ of I . We may assume the intervals of I and I^c are equally spaced on S^1 . Let g be the anti-clock wise rotation of S^1 by $\frac{2\pi}{\ell}$. Assume $I_{i+1} = g \cdot I_i$, $i = 1, \dots, 2\ell - 1$.

Denote by $\tilde{\rho} \in \text{End}(A'_{I^c})$ such that $\tilde{\rho}(A'_{I^c}) = A_I$. Such an endomorphism always exists since A_I and A'_{I^c} are hyperfinite type III₁ factors. Define an endomorphism $\rho \in \text{End}(A_I)$ by restricting $\tilde{\rho}$ to A_I , i.e. $\rho(x) = \tilde{\rho}(x)$ for any $x \in A_{I^c}$.

Since $\tilde{\rho}$ is an isomorphism from A'_{I^c} to A_{I^c} , the ring generated by $[\rho\lambda]$, as sectors of A_I is isomorphic to the ring generated by $[\lambda\tilde{\rho}]$ as sectors of A'_{I^c} .

Proposition 3.2.1. (1) *For any irreducible $\lambda \in \ddot{P}_+^m$, $[\lambda\tilde{\rho}]$ is irreducible, so is $[\rho\lambda]$;*
(2) *The ring generated by $[\lambda_{g_i}]$ for all $\lambda \in \ddot{P}_+^m$, $i = 1, 2, \dots, \ell$ is isomorphic to $\ddot{G}r_m^{\otimes \ell}$. The isomorphism φ is given by: $\varphi([\lambda_{g_i}]) = 1 \otimes \dots \otimes \lambda \otimes \dots \otimes 1$ where λ is on the i -th position and there are $\ell - 1$ 1's elsewhere. We shall identify $[\lambda_{g_i}]$ with its image under φ ;*

- (3) $[\lambda_{g_i} \mu_{g_j} \tilde{\rho}] = \Sigma_\nu N_{\lambda\mu}^\nu [\nu \tilde{\rho}]$, $[\rho \lambda_{g_i} \mu_{g_j}] = \Sigma_\nu N_{\lambda\mu}^\nu [\rho \nu]$;
- (4) $[\lambda_1 \otimes \dots \otimes \lambda_\ell] = [\mu_1 \otimes \dots \otimes \mu_\ell]$ if and only if $[\lambda_i] = [\mu_i]$, $1 \leq i \leq \ell$;

(5) If ρ has finite index, then $\bar{\rho}\rho \succ \Sigma_{\lambda_1, \dots, \lambda_\ell} N_{\lambda_1, \dots, \lambda_\ell}^1 \lambda_1 \otimes \lambda_2 \otimes \dots \otimes \lambda_\ell$, and $d_\rho \geq \frac{1}{S^{\ell-1}(\Lambda_0)}$. Here $N_{\lambda_1, \dots, \lambda_\ell}^1$ is the coefficient of id in the decomposition of $\lambda_1 \cdot \lambda_2 \cdots \lambda_\ell$.

Proof: (1) $\lambda\tilde{\rho}(A'_{I_c}) = \lambda(A_I) \subset A'_{I_c}$ is conjugate to the Jones-Wasserman subfactor $\pi_\lambda(A_I) \subset \pi_\lambda(A_{I_c})'$ by local equivalence. Since $\pi_\lambda(A_I) \subset \pi_\lambda(A_{I_c})'$ is irreducible if λ is irreducible (cf.[W2]). The second statement follows from the first one and the remark before the statement of Proposition 3.2.1.

(2) By using factorization (1) of Proposition 2.3.1, there exists an isomorphism $\psi : A_I \rightarrow A_{I_1} \hat{\otimes} A_{I_3} \hat{\otimes} \dots \hat{\otimes} A_{I_{2\ell-1}}$ such that

$$\psi(x_i) = 1 \hat{\otimes} \dots \hat{\otimes} x_i \hat{\otimes} \dots \otimes 1 \quad \text{if } x_i \in A_{I_{2i-1}}.$$

It is easy to see that $\psi \cdot \lambda_{g_i} \cdot \psi^{-1}$ becomes an endomorphism $1 \otimes \dots \otimes \lambda_{g_i} \otimes \dots \otimes 1$ on $A_{I_1} \hat{\otimes} A_{I_3} \hat{\otimes} \dots \otimes A_{I_{2\ell-1}}$.

(3) Let us recall that $\lambda_{g_i}(x) = \Gamma_\lambda(g_i)\lambda(x)\Gamma_\lambda(g_i)^*$, $\mu_{g_j}(x) = \Gamma_\mu(g_j)\mu(x)\Gamma_\mu(g_j)^*$, where $\Gamma_\lambda(g_i), \Gamma_\mu(g_j) \in A'_{I_c}$. Hence as sectors of A'_{I_c} , $[\lambda_{g_i}\mu_{g_j}\tilde{\rho}] = [\lambda\mu\tilde{\rho}]$. The first identity follows by the fact $[\lambda\mu] = \Sigma_\nu N_{\lambda\mu}^\nu[\nu]$ (See Theorem 2.2). The second identity follows from the first one and the remark before the statement of Proposition 3.2.1.

(4) Notice $[\lambda_1 \otimes \dots \otimes \lambda_\ell]$ is irreducible. If $[\lambda_1 \otimes \dots \otimes \lambda_\ell] = [\mu_1 \otimes \dots \otimes \mu_\ell]$, then $\lambda_i \cdot \bar{\mu}_1 \otimes \dots \otimes \lambda_\ell \cdot \bar{\mu}_\ell \succ 1 \otimes 1 \otimes \dots \otimes 1$ where 1 stands for the trivial sector.

But $\lambda_i \cdot \bar{\mu}_1 \otimes \dots \otimes \lambda_\ell \cdot \bar{\mu}_\ell = \Sigma_{\gamma_i} \prod_{i=1}^\ell N_{\lambda_i \bar{\mu}_i}^{\gamma_i} \gamma_1 \otimes \dots \otimes \gamma_\ell$. If we can show that $[\gamma_1 \otimes \dots \otimes \gamma_\ell] = [1 \otimes 1 \otimes \dots \otimes 1]$ if and only if $[\gamma_i] = [1]$, then it follows that $\prod_{i=1}^\ell N_{\lambda_i \bar{\mu}_i}^1 = 1$ and we obtain $[\lambda_i] = [\mu_i]$. So it is enough to show $[\gamma_1 \otimes \dots \otimes \gamma_\ell] = [1 \otimes \dots \otimes 1]$ implies $[\gamma_i] = [1]$.

Suppose $U \in A(I_1) \hat{\otimes} \dots \hat{\otimes} A(I_{2\ell-1})$ and $\gamma_1 \otimes \dots \otimes \gamma_\ell(x) = UxU^*$ for any $x \in A(I_1) \hat{\otimes} \dots \hat{\otimes} A(I_{2\ell-1})$. Then it follows that there exists a unitary operator $\tilde{U} : H_{\gamma_1} \otimes \dots \otimes H_{\gamma_\ell} \rightarrow H_0 \otimes H_0 \otimes \dots \otimes H_0$ which intertwines the action of

$$(\mathcal{L}_{I_1}G \vee \mathcal{L}_{I_1^c}G) \times (\mathcal{L}_{I_3}G \vee \mathcal{L}_{I_3^c}G) \times \dots \times (\mathcal{L}_{I_{2\ell-1}}G \vee \mathcal{L}_{I_{2\ell-1}^c}G)$$

on $H_{\gamma_1} \otimes \dots \otimes H_{\gamma_\ell}$ and $H_0 \otimes H_0 \otimes \dots \otimes H_0$.

Since $\pi_{\gamma_i}(\mathcal{L}_{I_{2i-1}}G \vee \mathcal{L}_{I_{2i-1}^c}G)$ is dense in $\pi_{\gamma_i}(\mathcal{L}G)$, by Theorem F of [W2] it follows that \tilde{U} intertwines the natural action of $\mathcal{L}G \times \dots \times \mathcal{L}G$ on $H_{\gamma_1} \otimes \dots \otimes H_{\gamma_\ell}$.

Notice $\overbrace{\mathcal{L}G \times \dots \times \mathcal{L}G}^\ell = \mathcal{L}(\overbrace{G \times G \cdots \times G}^\ell)$. For any $g \in \mathbf{G}$. Denote by $\pi_\gamma(g)$ (resp. $\pi_0(g)$) the intertwining action of \mathbf{G} on $H_{\gamma_1} \otimes \dots \otimes H_{\gamma_\ell}$ (resp. $H_0 \otimes \dots \otimes H_0$). We claim that $\tilde{U}\pi_\gamma(g)\tilde{U}^* = \pi_0(g)$. In fact, $\tilde{U}\pi_\gamma(g)\tilde{U}^*\pi_0(g)^*$ commutes with the action of $\mathcal{L}(G \times \dots \times G)$ on $H_0 \otimes \dots \otimes H_0$. The action of $\mathcal{L}(G \times \dots \times G)$ on $H_0 \otimes \dots \otimes H_0$ is irreducible. Since if $a \in \pi_0(\mathcal{L}G \times \dots \times G)'$, then

$$\begin{aligned} a &\in (A(I_1) \otimes \dots \otimes A(I_{2\ell-1}) \vee A(I_1^c) \otimes \dots \otimes A(I_{2\ell-1}^c))' \\ &= (B(H_0) \otimes \dots \otimes B(H_0))' = \mathbb{C}1. \end{aligned}$$

So $\tilde{U}\pi_\gamma(g)\tilde{U}^*\pi_0(g)^*$ is a scalar, and $g \rightarrow \tilde{U}\pi_\gamma(g)\tilde{U}^*\pi_0(g)^*$ is an abelian representation of \mathbf{G} which is necessarily trivial since \mathbf{G} is a perfect group (See the proof of Prop.2.2 in [GL1]).

It follows that the lowest energy states on $H_{\nu_1} \otimes \cdots \otimes H_{\nu_\ell}$ has the same energy as the lowest energy states of $H_0 \otimes H_0 \otimes \cdots \otimes H_0$, which is zero. This is impossible unless all γ_i 's are trivial representations of $SU(n)$, i.e.

$$[\gamma_i] = [1], \quad i = 1, \dots, \ell.$$

(5) By (2) and (3), we have

$$[\rho\lambda_1 \otimes \lambda_2 \otimes \cdots \otimes \lambda_\ell] = \sum_{\lambda_1 \cdots \lambda_\ell} N_{\lambda_1 \cdots \lambda_\ell}^\mu [\rho\mu]$$

where $N_{\lambda_1 \cdots \lambda_\ell}^\mu = \langle \lambda_1 \cdots \lambda_\ell, \mu \rangle$. If ρ has finite index, by Frobenius duality and (4), we have

$$\bar{\rho}\rho \succ \sum_{\lambda_1, \dots, \lambda_\ell} N_{\lambda_1 \cdots \lambda_\ell}^0 \lambda_1 \otimes \lambda_2 \otimes \cdots \otimes \lambda_\ell.$$

By the properties of statistical dimension, we have

$$d_\lambda \cdot d_\mu = \sum_{\gamma} N_{\lambda\mu}^\gamma d_\gamma.$$

We can use this property to calculate the following:

$$\begin{aligned} & \sum_{\lambda_1, \dots, \lambda_\ell} N_{\lambda_1 \cdots \lambda_\ell}^1 d_{\lambda_1} \cdots d_{\lambda_\ell} \\ &= \sum_{\lambda_1, \dots, \lambda_\ell} (N_{\lambda_1 \cdots \lambda_{\ell-1}}^{\lambda_\ell} d_{\lambda_\ell}) d_{\lambda_1} \cdots d_{\lambda_{\ell-1}} \\ &= \sum_{\lambda_1, \dots, \lambda_{\ell-1}} d_{\lambda_1}^2 \cdots d_{\lambda_{\ell-1}}^2 \\ &= \left(\sum_{\lambda_1} d_{\lambda_1}^2 \right)^{\ell-1} = \frac{1}{S^{2\ell-2}(\Lambda_0)} \end{aligned}$$

where we have also used

$$N_{\lambda_1 \cdots \lambda_\ell}^1 = N_{\lambda_1 \cdots \lambda_{\ell-1}}^{\lambda_\ell^*}$$

, $d_\lambda = d_{\lambda^*}$, and $\sum_\lambda d_\lambda^2 = \frac{1}{S^2(\Lambda_0)}$. So we have

$$d_\rho^2 \geq \frac{1}{S^{2\ell-2}(\Lambda_0)}$$

, i.e., $d_\rho \geq \frac{1}{S^{\ell-1}(\Lambda_0)}$. \square

Recall from the end of §2.2 that for $\mathcal{L}U(1)$ when n is even, the covariant representation π_i of the irreducible conformal precosheaf associated with $\mathcal{L}U(1)$ generates a ring isomorphic to the group ring of \mathbb{Z}_n . By exactly the same argument as the proof of Proposition 3.2.1, we see that Proposition 3.2.1 holds for the representations of $\mathcal{L}U(1)$ when n is even.

In this case, $\frac{1}{S(\Lambda_0)^2} = \sum_{i=0}^{n-1} d_i = n$ where $d_i = 1$ is the statistical dimension of each π_i .

From the same argument as in the proof of (5) of Proposition 3.2.1 we have

$$d(\pi_0(\mathcal{L}_I U(1)))'' \subset (\mathcal{L}_{I^c} U(1))' \geq n^{\frac{\ell-1}{2}}.$$

We shall prove a similar estimation when n is odd in the next section.

3.3. Odd n Case.

When n is odd, $\mathcal{L}T$ doesn't have local structure. In fact, it follows from the end of Section 2.1, $\pi_0((e^{if}, \lambda)(e^{ig}, \mu)(e^{if}, \lambda)^{-1}) = (-1)^{w(f)w(g)} \pi_0((e^{ig}, \mu))$ if e^{if}, e^{ig} have disjoint support. Our aim is to show, however, that the monodromy equation of Proposition 1.4.3 holds.

The vacuum representation F_0 decomposes into $F_0^{(0)} \oplus F_0^{(1)}$, where

$$F_0^{(0)} = \oplus_{d \equiv 0 \pmod{2n}} F_{0,d}, \quad F_0^{(1)} = \oplus_{d \equiv n \pmod{2n}} F_{0,d}.$$

Recall Γ acts as 1 on $F_0^{(0)}$ and -1 on $F_0^{(1)}$, and k acts as 1 on $F_0^{(0)}$ and i on $F_0^{(1)}$. Denote by $\mathcal{A}(I) = \pi_0(\mathcal{L}_I T)''$, $I \in \mathcal{I}$. Then we have twisted Haag-duality

$$\mathcal{A}(I)' = k^{-1} \mathcal{A}(I^c) k$$

which follows from Takesaki's devisage and geometric modular theory for fermions on the circle as in §15 of [W2].

Recall from §2 that an operator $A \in B(F_0)$ is called even (resp. odd) if $A = \Gamma A \Gamma$ (resp. $A = -\Gamma A \Gamma$). Every operator A decomposes uniquely into $A^+ + A^-$, where A^+ is even and A^- is odd. In fact $A^+ = \frac{A + \Gamma A \Gamma}{2}$, $A^- = \frac{A - \Gamma A \Gamma}{2}$.

For any $g \in \mathcal{D}$ which is the central extension of $\text{Diff}^+(S^1)$, we know from [PS] that the operator $\pi_i(g)$ preserves $F_{i,d}$, so $\pi_i(g)$ is an even operator.

It is easy to see $\mathcal{A}(I)$ satisfies A, B, C of §1.1. Let α be localized on I_1 . For any $J \in \mathcal{I}$, $\pi_0(\text{Ad}_{\alpha^i} \cdot \mathcal{L}_J T) = \pi_0(\mathcal{L}_J T) \cong \pi_i(\mathcal{L}_J T)$ is a type III₁ hyperfinite factor, $i = 0, 1, \dots, n-1$. So there exists a unitary operator $V_J \in \mathcal{U}(F_0)$ such that

$$\pi_0(\text{Ad}_{\alpha^i} \cdot x) = V_J^i \cdot \pi_0(x) V_J^{-i}$$

for any $x \in \mathcal{L}_J T$. Define:

$$\rho_J^i(y) = V_J^i \cdot y \cdot V_J^{-i}$$

for any $y \in \mathcal{A}(J) = \pi_0(\mathcal{L}_J T)''$. Notice $\rho_J^i(\mathcal{A}(J)) = \mathcal{A}(J)$ for any J , and $\rho_J^n(\mathcal{A}(J)) = W \cdot \mathcal{A}(J) \cdot W^*$, where $W = \pi_0(\beta) \in \mathcal{A}(I)$.

It is also clear from the definition that if $I \subset J$ and $x \in \mathcal{A}(I) \subset \mathcal{A}(J)$, then

$$\rho_J^i(x) = \rho_I^i(x)$$

Let us first show that $\rho_J^i \in \text{End}(\mathcal{A}(J))$ is not inner if $J \supset I_1$. Suppose $\pi_J^i(x) = UxU^*$ for any $x \in \mathcal{A}(J)$, and $U \in \mathcal{A}(J)$. Then $U^n x (U^*)^n = W^i x (W^i)^*$ for any $x \in \mathcal{A}(J)$. So $W^{-i} U^n \in \mathcal{A}(J) \cap \mathcal{A}(J)' = \mathbb{C}1$. It is easy to see that we have $\rho_J^i(\Gamma x \Gamma) = \Gamma \rho_J^i(x) \Gamma$. Since

$$\Gamma U \Gamma x \Gamma U^* \Gamma = \Gamma \rho_J^i(\Gamma x \Gamma) \Gamma = \rho_J^i(x)$$

$$\Gamma U \Gamma U^* \in A_J \cap A_J' = \mathbb{C}1.$$

, we must have $\Gamma U \Gamma = \pm U$. But $W^{-i} U^n \in \mathcal{A}(J) \cap \mathcal{A}(J)' = \mathbb{C}1$, W has winding number 1 mod 2 and n is an odd number, we conclude $\Gamma U \Gamma = (-1)^i U$. It follows that $\rho_{J^c}^i(x) = UxU^*$ for any $x \in \mathcal{A}(J^c)$. So $\pi_i(y) \cong \pi_0(\text{Ad}_{\alpha^i} y) = U\pi_0(y)U^*$ for any $y \in \mathcal{L}_J T \cup \mathcal{L}_{J^c} T$. Since $\pi_i(\mathcal{L}_J T) \vee \pi_i(\mathcal{L}_{J^c} T)$ generates $\pi_i(\mathcal{L}T)$ by Theorem F of [W2],

we have $\pi_i(y) \cong U\pi_0(y)U^*$ for any $y \in \mathcal{L}T$, a contradiction.

Fix $I_1 \in \mathcal{I}$. Choose an interval I_{ξ_1} such that $I_{\xi_1} \cap I_1 = \emptyset$. We define $W = \{g \in \mathbf{G} \mid I_1 \cup g.I, \text{ is a proper interval of } S^1 \text{ and } I_1 \cup g.I_1 \subsetneq S^1 - I_{\xi_1}\}$. For any $g \in W$, define $\Gamma_j(g) = \pi_j(g)\pi_0(g)^*$. As in §2.5, $\Gamma_j(g) \in \mathcal{A}((I \cup g.I)^c)'$, and since $\Gamma_j(g)$ is even, $\Gamma_j(g) \in \mathcal{A}(I_1 U g I_1)$. We shall define $U_{i,j}(g) = \rho_{I_0 \cup g I_0}^i(\Gamma_j(g))\pi_i(g)$. It is easy to check that if $g_1 \in W, g_2 \in W, g_1 g_2 \in W$, then

$$U_{i,j}(g_1 g_2) = U_{i,j}(g_1) U_{i,j}(g_2).$$

For any $g \in \mathbf{G}$, g can be written as $g = g_1 \cdots g_n$ since \mathbf{G} is simply connected. We define $U_{i,j}(g) = U_{i,j}(g_1) \cdots U_{i,j}(g_n)$. The following Proposition summarizes the properties of $U_{i,j}(g)$.

Proposition 3.3.1.. (i) $U_{i,j}(g)$ is well defined, i.e. if $g = g_1 \cdots g_n = h_1 \cdots h_m$ with $g_i, h_j \in W$, then $U_{i,j}(g_1) \cdots U_{i,j}(g_n) = U_{i,j}(h_1) \cdots U_{i,j}(h_m)$.

(ii) For any $J \in \mathcal{I}$, $x \in \mathcal{A}(J)$, we have

$$U_{i,j}(g) \rho_J^{i+j}(x) U_{i,j}(g)^* = \rho_{gJ}^{i+j}(\pi_0(g)x\pi_0(g)^*).$$

Proof: (i) It is sufficient to prove that if $g_1 \cdots g_n = e$, with $g_i \in W$, then $U_{i,j}(g_1) \cdots U_{i,j}(g_n) = \text{id}$. Since \mathbf{G} is a simply connected Lie group, we can find a path $\gamma_0 : [0, 1] \rightarrow \mathbf{G}$ with the following properties:

- (a) $\gamma_0\left(\frac{k}{n}\right) = g_1 \cdots g_k, 1 \leq k \leq n$, and $\gamma_0(0) = e$;
- (b) If $\frac{k-1}{n} \leq S \leq \frac{k}{n}, k = 1, \dots, n$; $\gamma_0(S) = g_1 \cdots g_{k-1} \cdot \gamma_k(S)$, where $\gamma_k(S)$ has the property:

$$\gamma_k\left(\frac{k-1}{n}\right) = e, \quad \gamma_k\left(\frac{k}{n}\right) = g_i, \quad \gamma_k(S) \in W,$$

and $\gamma_k(S_2) = \gamma_k(S_1)\gamma_k(S_2 - S_1)$ for any $\frac{k-1}{n} \leq S_1 \leq S_2 \leq \frac{k}{n}$.

From the fact that \mathbf{G} is simply connected, we can find $\gamma : [0, 1]^2 \rightarrow \mathbf{G}$ such that

$$\gamma(0, S) = \gamma_0(S), \quad \gamma(1, S) = e.$$

Since $[0, 1]^2$ is a compact set, we can find $\delta_0 > 0$, such that if $x, y \in [0, 1]^2$, $\|x - y\| < \delta_0$, then $\gamma(x) \in \gamma(y) \cdot W$. Choose integer m with $mn > \frac{1}{\delta_0}$. Define $g_k^{(l)}(t) = \gamma\left(t, \frac{k-1}{n} + \frac{l-1}{mn}\right)^{-1} \cdot \gamma\left(t, \frac{k-1}{n} + \frac{l}{mn}\right)$, $0 \leq t \leq 1$. Then $g_k^{(l)}(t) \in W$. It follows from the property of $U_{i,j}$ above that

$$U_{i,j}(g_k) = \prod_{l=1}^m U_{i,j}(g_k^{(l)}(0)).$$

Define

$$U_{i,j}(g_k)(t) = \prod_{l=1}^m U_{i,j}(g_k^{(l)}(t))$$

where the order of the product is from the left to the right as j increases. Consider a function $U_{i,j}(t) : [0, 1] \rightarrow U(F_0)$ defined by:

$$U_{i,j}(t) = U_{i,j}(g_1)(t) \cdots U_{i,j}(g_n)(t).$$

Notice

$$\begin{aligned} U_{i,j}(0) &= U_{i,j}(g_1) \cdots U_{i,j}(g_n), \quad \text{and} \\ U_{i,j}(1) &= \text{id}. \end{aligned}$$

We shall show that $U_{i,j}(t)$ is a locally constant function of t . Fix $t_0 \in [0, 1]$, there exists a neighborhood N of t_0 in $[0, 1]$, such that, if $t \in N$, then: $g_k^{(l)}(t_0)^{-1} \cdot g_k^{(l)}(t)$, and

$$h \cdot g_k^{(l)}(t_0)^{-1} \cdot g_k^{(l)}(t) \cdot h^{-1} \in W$$

for any h which can be written as products of not more than mn $g_k^{(l)}(t_0)$'s. Moreover, we require not more than mn products of $h \cdot g_k^{(l)}(t_0)^{-1} \cdot g_k^{(l)}(t) \cdot h^{-1}$ belong to W .

To simplify our formula, we define: $g_k^{(l)}(t_0) = g_{kl}$, $g_k^{(l)}(t) = g'_{kl}$ and $g'_{kl} = g_{kl}b_{kl}$.

Notice that

$$\prod_{a=1}^{nm} g_a = \prod_{a=1}^{nm} g'_a = e$$

By using the property of $U_{i,j}(g)$ and our choices of neighborhood N of t_0 , we have:

$$\begin{aligned}
\prod_{a=1}^{nm} U_{i,j}(g'_a) &= \prod_{a=1}^{nm} U_{i,j}(g_a) U_{i,j}(b_a) \\
&= U_{i,j}(g_1 b_1 g_1^*) \cdots U_{i,j}(g_1 \cdots g_{mn} b_{mn} g_{mn}^* \cdots g_1^*) \prod_{a=1}^{nm} U_{i,j}(g_a) \\
&= U_{i,j}(e) \prod_{a=1}^{nm} U_{i,j}(g_a) \\
&= \prod_{a=1}^{nm} U_{i,j}(g_a).
\end{aligned}$$

Since $[0, 1]$ is connected, we have proved $U_{i,j}(g_1) \cdots U_{i,j}(g_n) = U_{i,j}(0) = U_{i,j}(1) = \text{id}$.

(ii) Because of (i) we just have to show the intertwining property for $g \in W$. There are two cases to consider:

- (a) If $g.J \cup I_1 \cup gI_1 \subset J_1 \subsetneq S^1$. Then $U_{i,j}(g) \rho_J^i \rho_J^j(x) U_{i,j}(g)^* = \rho_{J_1}^i(\rho_{gJ}^j(\alpha_g x)) = \rho_{gJ}^{i+j}(\alpha_g \cdot x)$ where $x \in \mathcal{A}(J)$.
- (b) If $g.J \cup I_1 \cup gI_1$ covers S^1 , assume $g.J = I_2 \cup I_3 \cup I_4$ with $I_2 \subset (I_1 \cup SI_1)^c$ and $I_3 \cup I_4 = g.J \cap (I_1 \cup gI_1)$.

If $\alpha_g \cdot \rho_J^j(x) \in \mathcal{A}(I_i)$, $i = 2, 3, 4$, then $\rho_J^j(x) \in \mathcal{A}(g^{-1} \cdot I_i) \cap \mathcal{A}(J) = \mathcal{A}(g^{-1} \cdot I_i)$. Hence $x = \rho_J^{-j} \rho_J^j(x) \in \rho_J^{-j}(\mathcal{A}(g^{-1} \cdot I_i)) \subset \mathcal{A}(g^{-1} \cdot I_i)$, and $\alpha_g \cdot x \in \mathcal{A}(I_i)$.

If $\alpha_g \cdot \rho_J^j(x) \in \mathcal{A}(I_2)$, then $U_{i,j}(g) \rho_J^{i+j}(x) U_{i,j}(g)^* = \Gamma^i \alpha_g \cdot \rho_J^j(x) \cdot \Gamma^i = \rho_J^{i+j}(\alpha_g \cdot x)$.

If $\alpha_g \cdot \rho_J^j(x) \in \mathcal{A}(I_i)$, $i = 3$ or 4 , then $U_{i,j}(g) \rho_J^{i+j}(x) U_{i,j}(g)^* = \rho_{I_0 \cup gI_0}^i(\rho_{gJ}^j(\alpha_g \cdot x)) = \rho_{I_0 \cup gI_0}^{i+j}(\alpha_g \cdot x) = \rho_{gJ}^{i+j}(\alpha_g \cdot x)$. Since $\mathcal{A}(J) = \mathcal{A}(g^{-1}I_2) \vee \mathcal{A}(g^{-1}I_3) \vee \mathcal{A}(g^{-1}I_4)$, the proof is complete. \square

Because of (ii) of Proposition 3.3.1, $U_{i,j}(g)$ intertwining the action of $\text{Ad}_{\alpha^{i+j}} \mathcal{LT}$ on F_0 . Recall $\pi_{i+j}(g)$ is the action of \mathbf{G} on F_0 that intertwines $\text{Ad}_{\alpha^{i+j}} \mathcal{LT}$. It follows that $U_{i,j}(g) \pi_{i+j}(g)^*$ commutes with the action of $\text{Ad}_{\alpha^{i+j}} \mathcal{L}\pi$ on F_0 , so it must be a scalar. So $g \mapsto U_{i,j}(g) \pi_{i+j}(g)^*$ is an abelian representation of \mathbf{G} . But \mathbf{G} is perfect, and any abelian representation of G must be trivial. We conclude that $U_{i,j}(g) = \pi_{i+j}(g)$.

We can define braiding operator σ_{ρ^k, ρ^j} is exactly the same way as in §1.4. By using Proposition 3.3.1, exactly the same argument as in the proof of Proposition 1.4.3 shows that the monodromy operator $\sigma_{\rho^j, \rho^k} \sigma_{\rho^k, \rho^j}$ is diagonalized by intertwiner $T_e : \rho^{k+j} \rightarrow \rho^k \rho^j$, i.e.:

$$T_e^* \sigma_{\rho^j, \rho^k} \sigma_{\rho^k, \rho^j} T_e = \frac{S_{\rho^{k+j}}}{S_{\rho^k} S_{\rho^j}}$$

Since $\rho^{k+j} = \rho^k \cdot \rho^j$, we can choose $T_e = \text{id}$. Recall from §2.2 $S_{\rho^j} = \pi_j(2\pi) = \exp(2\pi i \cdot \Delta_j)$, where Δ_j is given by $\Delta_j = \frac{j^2}{2n}$.

It follows that: $\sigma_{\rho^j, \rho^k} \sigma_{\rho^k, \rho^j} = \exp\left(2\pi i \cdot \frac{kj}{n}\right)$.

Now we are ready to prove:

Proposition 3.3.2. *Suppose I is $\ell - 1$ -disconnected, then:*

$$d(\pi_i(\mathcal{A}(I)) \subset k\pi_i(\mathcal{A}(I^c))'k^{-1}) \geq n^{\frac{\ell-1}{2}}, i = 0, 1, \dots, n-1$$

.

Proof: It is clear that we have

$$d(\pi_i(\mathcal{A}(I)) \subset k\pi_i(\mathcal{A}(I^c))'k^{-1}) = d(\mathcal{A}(I) \subset k\mathcal{A}(I^c)'k^{-1})$$

since $\pi_i(\mathcal{A}(I)) \cong \pi_0(\text{Ad}_{\alpha^i} \cdot \mathcal{A}(I)) = \rho^i(\mathcal{A}(I)) = \mathcal{A}(I)$, so we just have to show $d(\mathcal{A}(I) \subset k\mathcal{A}(I^c)'k^{-1}) \geq n^{\frac{\ell-1}{2}}$.

Let $\tilde{\rho} \in \text{End}(k\mathcal{A}(I^c)'k^{-1})$ with $\tilde{\rho}(k\mathcal{A}(I^c)'k^{-1}) = \mathcal{A}(I)$. Define $\delta(i_1, \dots, i_\ell) = \rho^{i_1} \dots \rho_g^{i_2} \dots \rho_{g^\ell}^{i_\ell}$. Recall $\Gamma_i(g)\rho_J^i(x) = \rho_{g,g,J}^i(x)\Gamma_i(g)$ and $\Gamma_i(g)$ is localized on any intergal $J \supset I_1 \cup g.I_1$. So in particular $\Gamma_i(g) \in k\mathcal{A}(I^c)'k^{-1}$. It follows that as endomorphisms of $k\mathcal{A}(I^c)'k^{-1}$,

$$[\delta\tilde{\rho}] = [\rho^{i_1+i_2+\dots+i_\ell}\tilde{\rho}].$$

Notice $\delta\tilde{\rho}$ is irreducible since $\delta\tilde{\rho}(k\mathcal{A}(I^c)'k^{-1}) = \tilde{\rho}(k\mathcal{A}(I^c)'k^{-1})$ and $\tilde{\rho}$ is irreducible.

We shall show that $\delta(i_1, \dots, i_\ell)$, $\delta(i'_1, \dots, i'_\ell)$ as endomorphisms of $\mathcal{A}(J)$, are unitarily equivalent, i.e. $[\delta(i_1, \dots, i_\ell)] = [\delta(i'_1, \dots, i'_\ell)]$ iff $(i_1, \dots, i_\ell) = (i'_1, \dots, i'_\ell)$.

It is enough to show if $(i_1, \dots, i_\ell) \neq (0, 0, \dots, 0)$, then $[\delta(i_1, \dots, i_\ell)]$ is not equal to the trivial sector. Since n is odd, by considering δ^2 instead of δ , we may assume i_1, \dots, i_ℓ are all even numbers. Assume $(i_1, \dots, i_\ell) \neq (0, 0, \dots, 0)$.

Suppose there exists $U \in \mathcal{A}(I)$, such that $\delta(x) = UxU^*$ for all $x \in \mathcal{A}(I)$. Since $\delta(\Gamma x \Gamma) = \Gamma \delta(x) \Gamma$, we have

$$\Gamma U \Gamma x \Gamma U^* \Gamma = \Gamma \delta(PxP)P = \delta(x).$$

So $\Gamma U \Gamma U^* \in \mathcal{A}(I) \cap \mathcal{A}(I)' = \mathbb{C}1$. We have $\Gamma U \Gamma = \pm U$. Since $\delta^n(x) = U^n x U^{-n} = WxW^*$ with $W \in \mathcal{A}(I)$, W even, we deduce that U is even by the fact that $U^n W^* \in \mathcal{A}(I) \cap \mathcal{A}(I)' = \mathbb{C}1$. From $(i_1, \dots, i_\ell) \neq (0, \dots, 0)$, without loss of generality, we may assume one $i_k \neq 0$, (for some $1 \leq k \leq \ell$). From $\delta(x) = UxU^*$ we have:

$$\Gamma_{i_k}(g)\rho^{i_k}(x)\Gamma_{i_k}^*(g) = \rho_{g^k}^{i_k}(x) = UxU^* \quad \text{for any } x \in \mathcal{A}(I_{2k-1}).$$

From which we find:

$$U^* \Gamma_{i_k}(g) \in \mathcal{A}(I_{2k-1})' \cap \mathcal{A}(I_2^c) \cap \mathcal{A}(I_4^c) \cap \dots \cap \mathcal{A}(I_{2\ell}^c).$$

Since $U, \Gamma_{i_k}(g)$ are both even operators, we have:

$$U^* \Gamma_{i_k}(g) \in \mathcal{A}(I_{2k-1}^c) \cap \mathcal{A}(I_2^c) \cap \mathcal{A}(I_4^c) \cap \dots \cap \mathcal{A}(I_{2\ell}^c).$$

So $\Gamma_{i_k}(g) \in \mathcal{A}(I) \vee \mathcal{A}(I_{2k-2}) \cup I_{2k-1} \cup I_{2k})' \cap \mathcal{A}(I_2^c) \cdots \cap \mathcal{A}(I_{2\ell}^c)$. If we choose $\bar{I}_{\xi_1} \subsetneq I_{2k}$, $\bar{I}_{\xi_2} \subsetneq I_{2k-2}$, then $\mathcal{A}(I) \subset \mathcal{A}(I_{\xi_1}^c) \cap \mathcal{A}(I_{\xi_2}^c)$, $\mathcal{A}((I_{2k-2} \cup I_{2k-1} \cup I_{2k})^c) \subset \mathcal{A}(I_{\xi_1}^c) \cap \mathcal{A}(I_{\xi_2}^c)$, so we have $\rho_{I_{\xi_1}^c}^j(\nu)^* \rho_{I_{\xi_2}^c}^j(\nu) = 1$ for any $0 \leq j \leq n-1$ where $\nu = \Gamma_{i_k}(g)$.

Comparing with the monodromy equation before the statement of Proposition 3.3.2, we derive the following equation:

$$\exp\left(2\pi i \cdot \frac{j - i_k}{n}\right) = 1 \quad \text{for any } 0 \leq j \leq n-1.$$

But the above equation is true if and only if $i_k = 0$, a contradiction.

Now denote by ρ_1 the restriction $\tilde{\rho}$ to A_I , then from above we have

$$[\rho_1 \delta] = [\rho_1 \rho^{i_1 + \cdots + i_\ell}]$$

and

$$[\delta(i_1, \dots, i_\ell)] = [\delta(i'_1, \dots, i'_\ell)] \quad \text{iff } (i_1, \dots, i_\ell) = (i'_1, \dots, i'_\ell).$$

Now we are ready to finish the proof of the proposition.

If $d_{\rho_1} = +\infty$, the proposition trivially holds. If $d_{\rho_1} < +\infty$, by using Frobenius duality we have:

$$\bar{\rho}_1 \rho_1 \succ \sum_{\substack{(I_1, \dots, i_\ell) \in Z_n^\ell \\ i_1 + \cdots + i_\ell \equiv 0 \pmod n}} \delta(i_1, \dots, i_\ell).$$

Notice all $\delta(i_1, \dots, i_\ell)$ have statistical dimension 1, so $d_{\rho_1} = d(A_I \subset kA'_{I^c} k^{-1}) \geq n^{\frac{\ell-1}{2}}$. \square

3.4. Level 1 case.

In this section, we shall show, by using conformal inclusion

$$LU(1) \times LSU(n) \subset LU(n)$$

that $\pi_0(L_I SU(n))' \subset \pi_0(L_{I^c} SU(n))'$ has index $n^{\ell-1}$ for a $\ell-1$ -disconnected interval. Here π_0 is the level 1 vacuum representation of $LSU(n)$. Let π be the basic representation of $LU(n)$ as in §2.

Consider the following analogue of basic construction:

$$\begin{aligned} \pi(L_I U(1) \times L_I SU(n))'' &\subset \pi(L_I U(n))'' = k\pi(L_{I^c} U(n))' k^{-1} \\ &\subset k\pi(L_{I^c} U(1) \times L_{I^c} SU(n))' k^{-1}. \end{aligned}$$

From (2) of Proposition 3.1.1 we conclude that the statistical dimension of

$$\pi(L_I U(1) \times L_I SU(n)) \subset k\pi(L_{I^c} U(1) \times L_{I^c} SU(n))' k^{-1}$$

is $(n)^\ell$. Then each reduced subfactor

$$\pi_i(L_I U(1)) \otimes \pi_{\Lambda_i}(L_I SU(n)) \subset k\pi_i(L_{I^c} U(1))' k^{-1} \otimes \pi_{\Lambda_i}(L_{I^c} SU(n))'$$

has finite statistical dimension.

By Theorem 2.2 and the end of §2.1 we have:

$$\begin{aligned} d(\pi_0(L_I U(1))'' \subset \pi_0(L_{I^c} U(1))') &= d(\pi_i(L_I U(1))'' \subset \pi_i(L_{I^c} U(1))') \\ d(\pi_{\Lambda_0}(L_I SU(n))'' \subset \pi_{\Lambda_0}(L_{I^c} SU(n))') &= d(\pi_{\Lambda_i}(L_I SU(n))'' \subset \pi_{\Lambda_i}(L_{I^c} SU(n))') \end{aligned}$$

. It follows from Proposition 2.1.2 that $nd'd = (n)^\ell$, where $d' = d(\pi_0(L_I U(1))'' \subset \pi_0(L_{I^c} U(1))')$, $d = d(\pi_{\Lambda_0}(L_I SU(n))'' \subset \pi_{\Lambda_0}(L_{I^c} SU(n))')$. By (5) of Proposition 3.2.1 and Proposition 3.3.2 and the end of §3.2, $d' \geq n^{\frac{\ell-1}{2}}$, $d \geq n^{\frac{\ell-1}{2}}$, we deduce that $d' = d = n^{\frac{\ell-1}{2}}$.

3.5 General Case.

We are now ready to prove our main theorem.

Theorem 3.5. *Let I be a $\ell - 1$ -disconnected interval of S^1 . $\lambda \in \ddot{P}_+^m$. Then:*

(1) *the Jones-Wassermann subfactor*

$$\pi_\lambda(L_I SU(n))'' \subset \pi_\lambda(L_{I^c} SU(n))'$$

is conjugate to $\rho\lambda(\mathcal{A}(I)) \subset \mathcal{A}(I)$. Here ρ is defined as in §3.2. The statistical dimension d of $\rho\lambda\mathcal{A}(I) \subset \mathcal{A}(I)$ is given by

$$d = \frac{\ddot{S}(\lambda)}{\ddot{S}(\ddot{\Lambda}_0)} \cdot \frac{1}{\ddot{S}^{\ell-1}(\ddot{\Lambda}_0)}$$

where, in accordance with the notation of §3.1, we have used \ddot{S} to denote the S -matrices of $LSU(n)$.

(2)

$$[\bar{\rho}\rho] = \sum_{\lambda_1 \dots \lambda_\ell \in \ddot{P}_+^m} N_{\lambda_1, \dots, \lambda_\ell}^1 [\lambda_1 \otimes \lambda_2 \otimes \dots \otimes \lambda_\ell]$$

where $N_{\lambda_1 \dots \lambda_\ell}^1$ is the coefficient of id in the decomposition of $\lambda_1 \cdot \lambda_2 \cdot \dots \cdot \lambda_\ell$ and $\lambda_1 \otimes \lambda_2 \otimes \dots \otimes \lambda_\ell$ is understood as (2) of Proposition 3.2.1. Hence all the Jones-Wassermann subfactors are finite depth subfactors.

Proof: Let us consider the conformal inclusion:

$$LSU(m) \times LSU(n) \subset LSU(mn).$$

As in §3.1, we shall denote the S -matrices of $LSU(m)_n$ (resp. $LSU(n)_m$) by $\dot{S}(\ddot{S})$. Let π^0 be the vacuum representation of $LSU(nm)$. We shall denote the statistical

dimension of the Jones-Wassermann subfactor for the vacuum representation of $LSU(m)$ (resp. $LSU(n)$) by $d_{\rho'}$ (resp. d_{ρ}).

To save some writing we shall use \dot{S}_{00} (resp. \ddot{S}_{00}) to denote $\dot{S}(\dot{\Lambda}_0)$ (resp. $\ddot{S}(\ddot{\Lambda}_0)$). Let us consider the following inclusions:

$$\begin{aligned} \pi^0(L_I SU(m) \times L_I SU(n)) &\subset \pi^0(L_I SU(nm))' \subset \pi^0(L_{I^c} SU(nm))' \\ &\subset \pi^0(L_{I^c} SU(m) \times L_{I^c} SU(n))'. \end{aligned}$$

It follows from §3.4 and (1) of Proposition 3.1.1 that the statistical dimension of

$$\pi^0(L_I SU(m) \times L_I SU(n)) \subset \pi^0(L_{I^c} SU(n) \times L_{I^c} SU(n))'$$

is

$$(nm)^{\frac{\ell-1}{2}} \cdot \frac{1}{n^{\ell}} \cdot \frac{1}{\ddot{S}_{00}^{2\ell}}.$$

From this we obtain an equation:

$$(nm)^{\frac{\ell-1}{2}} \cdot \frac{1}{n^{\ell} \cdot \ddot{S}_{00}^{2\ell}} = d_{\rho'} d_{\rho} \cdot \frac{1}{n \ddot{S}_{00}^2}$$

. Recall from (3) of Lemma 3.1.1

$$\dot{S}_{00} = \sqrt{\frac{n}{m}} \ddot{S}_{00}.$$

So we have

$$\begin{aligned} d_{\rho'} \cdot d_{\rho} &= \frac{1}{\dot{S}_{00}^{\ell-1} \left(\sqrt{\frac{n}{m}} \dot{S}_{00} \right)^{\ell-1}} \\ &= \frac{1}{\dot{S}_{00}^{\ell-1}} \cdot \frac{1}{\ddot{S}_{00}^{\ell-1}}. \end{aligned}$$

From Proposition 3.2.1, we know

$$d_{\rho'} \geq \frac{1}{\dot{S}_{00}^{\ell-1}}, \quad d_{\rho} \geq \frac{1}{\ddot{S}_{00}^{\ell-1}}.$$

So we have

$$d_{\rho'} = \frac{1}{\dot{S}_{00}^{\ell-1}}, \quad d_{\rho} = \frac{1}{\ddot{S}_{00}^{\ell-1}}.$$

The rest of the theorem follows immediately from Proposition 3.2.1. \square

Recall from (3) of Proposition 3.2.1 that

$$[\rho \lambda_1 \otimes \cdots \otimes \lambda_{\ell}] = \sum_{\lambda_{\ell+1}} N_{\lambda_1 \dots \lambda_{\ell}}^{\lambda_{\ell+1}} [\rho \lambda_{\ell+1}]$$

where $[\rho \lambda_{\ell+1}]$ is irreducible.

This, combined with (2) of Theorem 3.4, determines the decompositions of $[(\bar{\rho}\rho)^n]$, $[(\bar{\rho}\rho)^n \bar{\rho}]$ completely in terms of the ring structure of $\check{G}r_m$ which is also completely determined (See §2.2). So the dual principal graph of the Jones-Wassermann subfactor

$$\pi_\lambda(L_I SU(n))'' \subset \pi_\lambda(L_{I^c} SU(n))'$$

is determined.

Notice if $\ell = 2$, by (2) of Theorem 3.4 d_ρ is equal to $\frac{1}{\check{S}_{00}}$ which is precisely the quantum 3-manifold invariant of type A_{n-1} evaluated on $S^2 \times S^1$. (See [To].)

§4. CONCLUSIONS

In this paper we have shown that all Jones-Wassermann subfactors, for disjoint intervals from loop groups of type A are of finite depth. The index value and the dual principle graphs of these subfactors are completely determined.

It is worth mentioning that the square root of the index of the Jones-Wassermann subfactor, which is obtained if we take the vacuum representation and choose the interval I to be 1-disconnected, is the same as quantum 3-manifold invariant of type A evaluated on $S^2 \times S^1$. It has been suggested in [W1] that the Jones-Wassermann subfactors for disjoint intervals should be related to higher genus conformal field theories. It will be very interesting, for e.g., to explain the coincidence above more directly along these lines.

Our methods of using conformal inclusions, in principle, apply to any LG where G is a classical simple compact Lie group. However, the case when G is exceptional seems to be a challenging question since in this case the conformal inclusions are not available for general level case.

References

- [B] D.Buchholz, C.D'Antoni and K.Fredenhagen, *local factorizations. The universal structure of local algebras*, Comm.Math.Phys., **111**, 123-135 (1987).
- [Boer] K.Fredenhagen, K.-H.Rehren and B.Schroer ,
Superselection sectors with braid group statistics and exchange algebras. II, Rev. Math. Phys. Special issue (1992), 113-157.
- [Fre] K.Fredenhagen, *Generalizations of the theory of superselection sectors*, in "The algebraic theory of superselection sectors", D.Kastler ed., World Scientific, Singapore 1990.
- [Fro] J. Frohlich and F. Gabbiani, *Operator algebras and CFT*, Comm. Math. Phys., **155**, 569-640 (1993).
- [GL1] D.Guido and R.Longo, *The Conformal Spin and Statistics Theorem*, hep-th 9505059.
- [GL2] D.Guido and R.Longo, *Relativistic invariance and charge conjugation in quantum field theory*, Comm.Math.Phys., **148**, 521-551
- [GL3] D.Guido and R.Longo, *An Algebraic Spin and Statistics Theorem*, to appear in Comm.Math.Phys.
- [Itz] D. Altschuler, M. Bauer and C. Itzykson, *The branching rules of conformal embeddings*, Comm.Math.Phys., **132**, 349-364 (1990).
- [K1] V. G. Kac and M. Wakimoto, *Modular and conformal invariance constraints in representation theory of affine algebras*, Advances in Math., **70**, 156-234 (1988).
- [K2] V. G. Kac, *Infinite dimensional algebras*, 3rd Edition, Cambridge University Press, 1990.
- [L1] R. Longo, *Minimal index and braided subfactors*, J. Funct. Anal., **109**, 98-112 (1992).
- [L2] R. Longo, *Duality for Hopf algebras and for subfactors*, I, Comm. Math. Phys., **159**, 133-150 (1994).
- [L3] R. Longo, *Index of subfactors and statistics of quantum fields*, I, Comm. Math. Phys., **126**, 217-247 (1989).
- [L4] R. Longo, *Index of subfactors and statistics of quantum fields*, II, Comm. Math. Phys., **130**, 285-309 (1990).
- [L5] R. Long, *Proceedings of International Congress of Mathematicians*, 1281-1291 (1994).
- [PS] A. Pressley and G. Segal, *Loop groups*, Clarendon Press, 1986.
- [Tu] V.G.Turaev, *Quantum Invariants of Knots and 3-Manifolds*,
Walter de Gruyter, Berlin, New York 1994.
- [W1] A. Wassermann, *Proceedings of International Congress of Mathematicians*, 966-979 (1994).
- [W2] A. Wassermann, *Operator algebras and Conformal field theories III*, to appear in Inv.Math.
- [W3] *Operator algebras and CFT*, preliminary notes.
- [X] F.Xu, *New Braided Endomorphisms from Conformal Inclusions*, submitted

to Comm.Math.Phys. 1996.

- [Y] S. Yamagami, *A note on Ocneanu's approach to Jones index theory*, Internat. J. Math., **4**, 859-871 (1993).

DEPARTMENT OF MATHEMATICS, UCLA, CA 90024

E-mail address: `xufeng@math.ucla.edu`